

"Par BRST Complex and quantised Hamiltonian Reduction" ①  
References Working seminar, Univ. Melbourne 16.05.2016.

T. Arakawa, arXiv:1211.7124, Rationality of  
W-algebras: Principal Nilpotent cases

T. Arakawa, arXiv: math-ph/0405015, Representation  
theory of supercentral algebras and the  
Kac-Roan-Wakimoto conjecture.

V. Kac, S. Roan and M. Wakimoto, arXiv: math-ph/0302015  
Quantum reduction for Affine Superalgebras

The universal affine vertex algebra  $V^k(\mathfrak{g})$

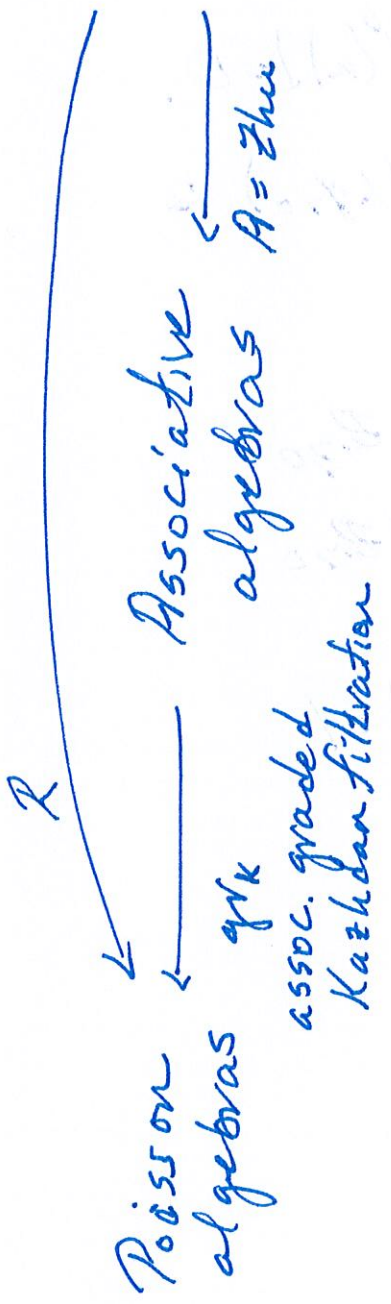
$$V^k(\mathfrak{g}) = U(\mathfrak{g}[t, t^{-1}] + \mathbb{C}K) \otimes U(\mathfrak{g}[t] \otimes \mathbb{C}K) \mathbb{C}$$

with  $x t^m \mathfrak{g} = 0$ , for  $x \in \mathfrak{g}$ ,  $m \in \mathbb{Z}_{\leq 0}$ , and  
 $K \cdot \mathfrak{g} = k \cdot \mathfrak{g}$ .

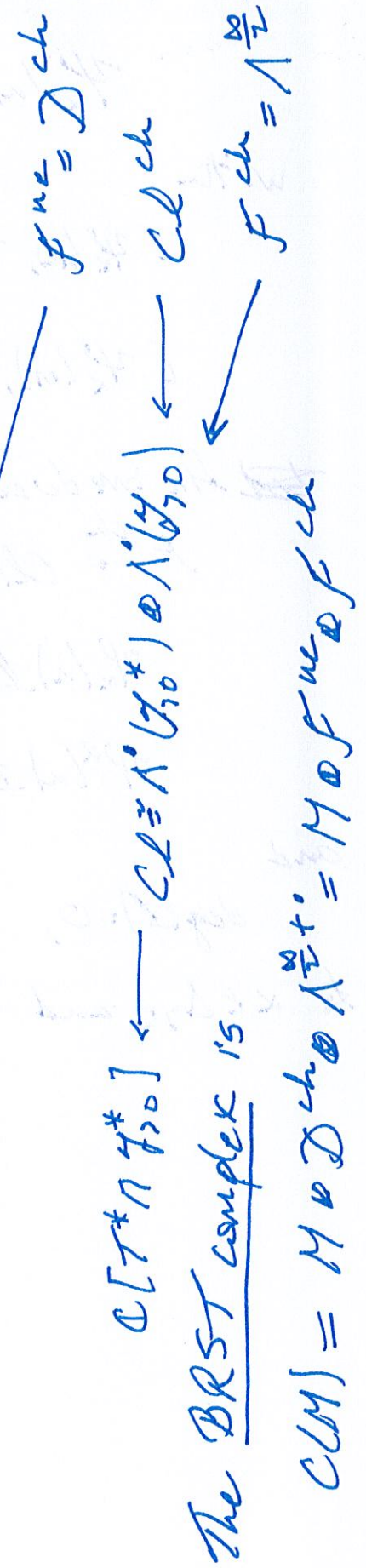
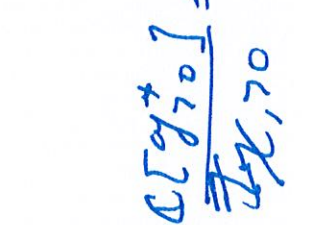
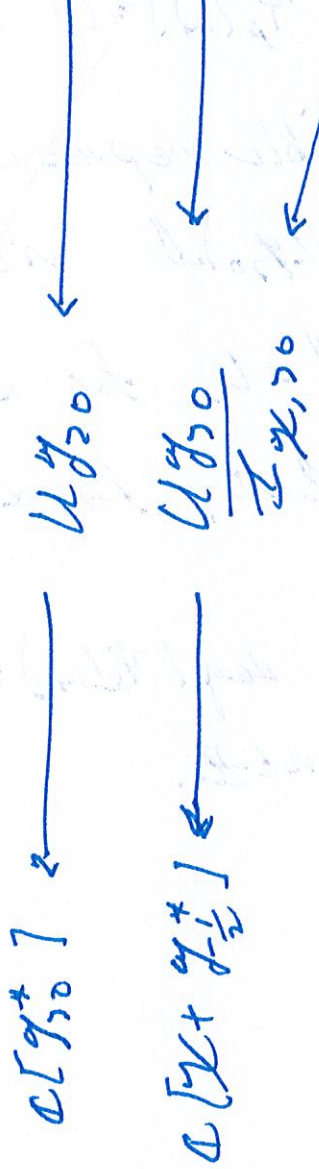
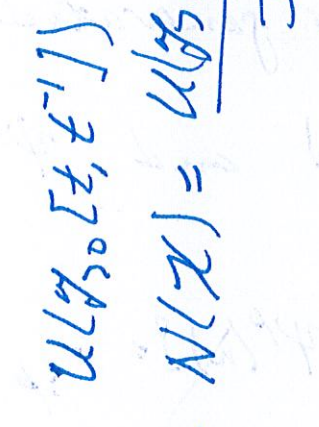
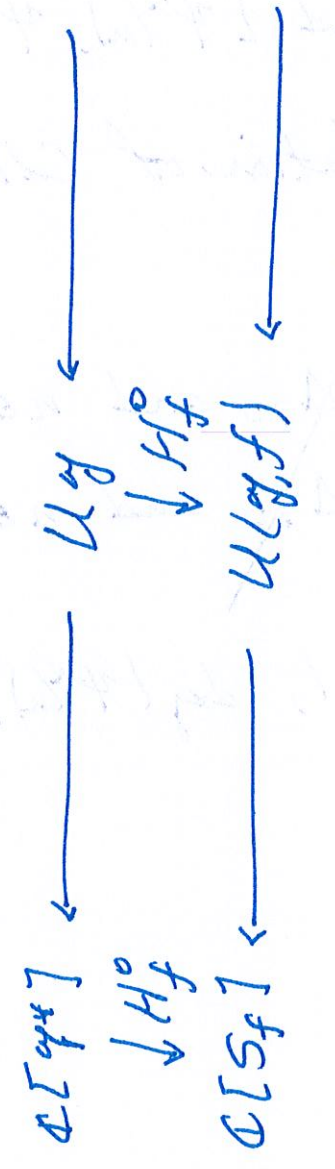
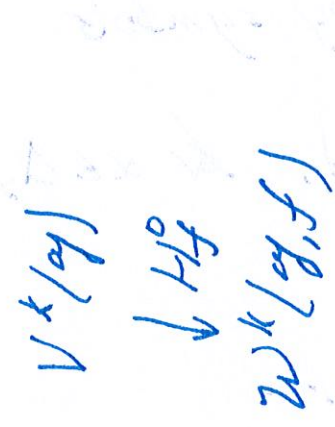
and  $\deg(x t^n) = -n$ ,  $\deg(\mathbb{C}K) = 0$

and  $Y(\cdot, z): \mathfrak{g} \rightarrow \mathfrak{g}[[z, z^{-1}]]$  given by

$$Y(x, z) = x(z) = \sum_{n \in \mathbb{Z}} (x t^n) z^{-n-1}$$



Vertex algebras



$$CM = M \otimes D^{ch} \otimes \Lambda^{\infty+} = M \otimes F_{ne} \otimes F^{ch}$$

Data

$\mathfrak{g}$  is fin. dim'l reductive Lie algebra.

(1) :  $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$       symm. bilinear nondeg  
 $(x, y) \mapsto (x|y)$       ad-invariant.

Let

$f \in \mathfrak{g}$  be nilpotent,  $\{e, f, h\}$  an  $\mathfrak{sl}_2$ -triple

$\mathcal{X} = \nu(f)$  where  $\nu: \mathfrak{g} \rightarrow \mathfrak{g}^*$   
 $x \mapsto (x| \cdot)$

Then

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j \quad \text{and} \quad \mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$$

Write

$$R_{>0} = \{ \alpha \in R \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{>0} \}$$

$$R_{\frac{1}{2}} = \{ \alpha \in R \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\frac{1}{2}} \}.$$

The algebra  $N(\chi)$  and

the neutral free superfermions  $F^{ne}$

$$\chi: \mathbb{Z}_2[t, t^{-1}] \longrightarrow \mathbb{C}$$

$$ut^m \longmapsto (f/da) \delta_{m,-1}$$

Then

$$I_\chi = U(\mathbb{Z}_2[t, t^{-1}]) \ker \chi$$

is a two sided ideal of  $U(\mathbb{Z}_2[t, t^{-1}])$

$$N(\chi) = \frac{U(\mathbb{Z}_2[t, t^{-1}])}{I_\chi}$$

An irreducible representation of  $N(\chi)$  is

$$F^{ne} = N(\chi) \otimes_{\mathbb{C}} \chi \quad \text{with}$$

$$ut^n \otimes_{\mathbb{C}} \chi = 0 \quad \text{for } u \in \mathbb{Z}_2 \text{ and } n \in \mathbb{Z}_{\neq 0}$$

Working seminar 16.05.2016  
ARom (4)

The algebra  $Cl^{ch}$  and

the charged free superfermions  $F^{ch}$

$Cl^{ch} = Cl(\mathbb{Z}_0[t, t^{-1}])$  is generated by

symbols  $\psi_\alpha(n)$  and  $\psi^\alpha(n)$ ,  $\alpha \in \mathbb{R}_{>0}$  and  $n \in \mathbb{Z}$

with relations

$$[\psi_\alpha(m), \psi_\beta(n)] = 0 \quad \text{and} \quad [\psi^\alpha(m), \psi^\beta(n)] = 0$$

$$[\psi_\alpha(m), \psi^\beta(n)] = \delta_{\alpha, \beta} \delta_{m, -n}.$$

An irreducible representation of  $Cl^{ch}$  is

$$F^{ch} = Cl^{ch} \underline{\mathbb{1}} \quad \text{with}$$

$$\psi_\alpha(n) \underline{\mathbb{1}} = 0, \quad \text{for } \alpha \in \mathbb{R}_{>0} \text{ and } n \in \mathbb{Z}_{>0}$$

$$\psi^\alpha(n) \underline{\mathbb{1}} = 0, \quad \text{for } \alpha \in \mathbb{R}_{>0} \text{ and } n \in \mathbb{Z}_{>0}.$$

and

$$\deg(\underline{\mathbb{1}}) = 0, \quad \deg(\psi_\alpha(n)) = -1, \quad \deg(\psi^\alpha(n)) = 1.$$

for  $\alpha \in \mathbb{R}_{>0}$  and  $n \in \mathbb{Z}$ .

Working seminar  
16.05.2016  
A. Ram

Associative case: Construction of  $\mathcal{D}$  (5)

$$\mathcal{I}_{\gamma_0, \kappa} = \sum_{x \in \gamma_{\gamma_0}} U(\gamma_{\gamma_0})(x - \kappa(x))$$

is a two sided ideal of  $U(\gamma_{\gamma_0})$ .

$$\mathcal{D} = \frac{U(\gamma_{\gamma_0})}{\mathcal{I}_{\gamma_0, \kappa}}$$

Then  $\mathcal{D}$  is a Heisenberg algebra of rank  $\frac{1}{2} \dim(\gamma_{\frac{\gamma_0}{2}})$ .

Associative case: Construction of  $\mathcal{Cl}$

$$\mathcal{Cl} \cong \wedge^*(\gamma_{\gamma_0}^*) \otimes \wedge^*(\gamma_{\gamma_0})$$

is a vector space isomorphism where

$\mathcal{Cl}$  is the Clifford algebra of  $\gamma_{\gamma_0} \oplus \gamma_{\gamma_0}^*$  with respect to the bilinear form given by

$$(x_1 + y_1, x_2 + y_2) = \varphi_1(x_2) + \varphi_2(x_1)$$

Then

$$\mathfrak{g}_K \mathcal{D} \cong \mathcal{C}[\mathcal{X} + \nu(\gamma_{\frac{\gamma_0}{2}})]$$

$$\mathfrak{g}_K \mathcal{Cl} \cong \mathcal{C}[\mathcal{T}^* \Pi \gamma_{\gamma_0}^*]$$

as Poisson algebras.

Poisson case: Construction of  $\bar{D}$

Let  $\bar{I}_{\gamma_0, \chi}$  be the Poisson ideal of  $\mathbb{C}[\mathfrak{g}_{\gamma_0}^*]$  generated by  $\{x - \chi(x) \mid x \in \mathfrak{g}_{\gamma_0, 1}\}$ .

$\chi + v(\mathfrak{g}_{\gamma_0, 1})$  is an affine subspace of  $\mathfrak{g}_{\gamma_0}^*$  and

$$\bar{D} = \frac{\mathbb{C}[\mathfrak{g}_{\gamma_0}^*]}{\bar{I}_{\gamma_0, \chi}} = \mathbb{C}[\chi + v(\mathfrak{g}_{\gamma_0, 1})]$$

Poisson case: Construction of  $\bar{\mathcal{L}}$

$\Pi \mathfrak{g}_{\gamma_0}^*$  is  $\mathfrak{g}_{\gamma_0}^*$  as a purely odd vector space

$T^* \Pi \mathfrak{g}_{\gamma_0}^*$  is the cotangent bundle

$$\bar{\mathcal{L}} = \mathbb{C}[T^* \Pi \mathfrak{g}_{\gamma_0}^*] = \Lambda^0(\mathfrak{g}_{\gamma_0}^* \oplus \mathfrak{g}_{\gamma_0}) = \Lambda^0(\mathfrak{g}_{\gamma_0}^*) \otimes \Lambda^0(\mathfrak{g}_{\gamma_0})$$

then

$$\bar{\mathcal{L}}(M) = M \otimes \bar{D} \otimes \bar{\mathcal{L}}.$$

Homework:

- (1) Work out normal ordering and PBW.
- (2) Virasoro normalization from Green-Schwarz-Witten
- (3) Lie algebra cohomology and Finkel-Grojanowski-Teleman
- (4) Work through §7 of Kac-Rain-Wakimoto.
- (5) smooth  $\hat{\mathfrak{g}}$ -modules and restricted  $\hat{\mathfrak{g}}$ -modules.

A  ~~$\hat{\mathfrak{g}}$ -module~~ smooth  $\hat{\mathfrak{g}}$ -module, <sup>or restricted  $\hat{\mathfrak{g}}$ -module,</sup> is a  $\hat{\mathfrak{g}}$ -module  $M$  such that

$$(xt^n)_m = 0, \text{ for } \cancel{m} x \in \mathfrak{g}, m \in M \text{ and } t \text{ large enough.}$$