"The BRST Complex and quantised Hamiltonian Reduction" (0)

<u>References</u> Working seminar, Univ. Melbourne 16.05.2016.

T. Arakawa, arXiv:1211.7124, Rationality of W-algebras: Principal Nilpotent cases

T. Arakawa, arXiv: math-ph/0405015, Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture.

V. Kac, S. Roan and M. Wakimoto, arXiv: math-ph/0302015 Quantum reduction for Affine Superalgebras

<u>The universal affine vertex algebra</u> $V^k(\mathfrak{g})$

$$V^k(\mathfrak{g}) = U(\mathfrak{g}[t,t^{-1}] + \mathbb{C}K) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}$$

with
$$x t^m \mathbb{1} = 0, \quad \text{for } x \in \mathfrak{g},\ m \in \mathbb{Z}_{\geq 0}, \quad \text{and}$$
$$K \cdot \mathbb{1} = k \mathbb{1}.$$

and
$$\deg(x t^n) = -n, \quad \deg(\mathbb{1}) = 0$$

and $Y(\cdot, z) : \mathfrak{g} \longrightarrow \mathfrak{g}[[z, z^{-1}]]$ given by

$$Y(x,z) = x(z) = \sum_{n \in \mathbb{Z}} (x t^n) z^{-n-1}.$$

$$
\begin{tikzcd}
\text{Poisson algebras} & \text{Associative algebras} \arrow[l, "\mathrm{gr}_K"'] & \text{Vertex algebras} \arrow[l, "A = \mathrm{Zhu}"'] \arrow[ll, bend right, "R"'] \\
\mathbb{C}[\mathfrak{g}^*] \arrow[d, "H_f^0"] & U\mathfrak{g} \arrow[l] \arrow[d, "H_f^0"] & V^k(\mathfrak{g}) \arrow[l] \arrow[d, "H_f^0"] \\
\mathbb{C}[S_f] & U(\mathfrak{g},f) \arrow[l] & W^k(\mathfrak{g},f) \arrow[l] \\
\mathbb{C}[\mathfrak{g}^*_{>0}] & U\mathfrak{g}_{>0} \arrow[l] & U(\mathfrak{g}_{>0}[t,t^{-1}]) \arrow[l] \\
\frac{\mathbb{C}[\mathfrak{g}^*_{>0}]}{I_{\chi,>0}} = \mathbb{C}[\chi + \mathfrak{g}^*_{-\frac{1}{2}}] & \frac{U\mathfrak{g}_{>0}}{I_{\chi,>0}} \arrow[l] & N(\chi) = \frac{U(\mathfrak{g}_{>0}[t,t^{-1}])}{I_\chi} \arrow[l] \\
 & & \mathcal{F}^{ne} = \mathcal{D}^{ch} \arrow[ul] \\
\mathbb{C}[T^*\Pi\mathfrak{g}^*_{>0}] & Cl = \Lambda^\bullet(\mathfrak{g}^*_{>0}) \otimes \Lambda^\bullet(\mathfrak{g}_{>0}) \arrow[l] & Cl^{ch} \arrow[l] \\
 & & \mathcal{F}^{ch} = \Lambda^{\frac{\infty}{2}+\bullet} \arrow[ul]
\end{tikzcd}
$$

assoc. graded
Kazhdan filtration

The BRST complex is

$$C(M) = M \otimes \mathcal{D}^{ch} \otimes \Lambda^{\frac{\infty}{2}+\bullet} = M \otimes \mathcal{F}^{ne} \otimes \mathcal{F}^{ch}$$

Data

$\mathring{\mathfrak{g}}$ is fin. dim'l reductive Lie algebra.

$(\mid) : \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \to \mathbb{C}$ symm. bilinear nondeg

$(x, y) \mapsto (x|y)$ ad-invariant.

Let

$f \in \mathring{\mathfrak{g}}$ be nilpotent, $\{e, f, h\}$ an '$\mathfrak{sl}_2$-triple'

$\chi = \nu(f)$ where $\nu : \mathring{\mathfrak{g}} \to \mathring{\mathfrak{g}}^*$

$x \mapsto (x| \;)$

Then

$$\mathring{\mathfrak{g}} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathring{\mathfrak{g}}_j \text{ and } \mathring{\mathfrak{g}} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathring{\mathfrak{g}}_\alpha \right)$$

Write

$$R_{>0} = \{ \alpha \in R \mid \mathring{\mathfrak{g}}_\alpha \subseteq \mathfrak{g}_{>0} \}$$

$$R_{\frac{1}{2}} = \{ \alpha \in R \mid \mathring{\mathfrak{g}}_\alpha \subseteq \mathring{\mathfrak{g}}_{\frac{1}{2}} \}.$$

The algebra $N(\chi)$ and the neutral free superfermions $\mathcal{F}^{ne}$

$$\chi : \mathfrak{g}_{\geq 1}[t,t^{-1}] \longrightarrow \mathbb{C}$$
$$ut^m \longmapsto (f|u)\,\delta_{m,-1}$$

Then

$$I_\chi = U(\mathfrak{g}_{>0}[t,t^{-1}])\ker\chi$$

is a two sided ideal of $U(\mathfrak{g}_{\geq 0}[t,t^{-1}])$

$$N(\chi) = \frac{U(\mathfrak{g}_{>0}[t,t^{-1}])}{I_\chi}$$

An irreducible representation of $N(\chi)$ is

$$\mathcal{F}^{ne} = N(\chi)\mathbb{1}_\chi \quad \text{with}$$

$$ut^n\mathbb{1}_\chi = 0 \quad \text{for } u \in \mathfrak{g}_{\frac{1}{2}} \text{ and } n \in \mathbb{Z}_{\geq 0}$$

The algebra Cl^{ch} and the charged free superfermions $\mathcal{F}^{ch}$

$Cl^{ch} = Cl(\mathfrak{h}_{>0}[t,t^{-1}])$ is generated by symbols $\psi_\alpha(n)$ and $\psi^\alpha(n)$, $\alpha \in R_{>0}$ and $n \in \mathbb{Z}$ with relations

$$[\psi_\alpha(m), \psi_\beta(n)] = 0 \quad \text{and} \quad [\psi^\alpha(m), \psi^\beta(n)] = 0$$

$$[\psi_\alpha(m), \psi^\beta(n)] = \delta_{\alpha,\beta}\,\delta_{m,-n}.$$

An irreducible representation of Cl^{ch} is

$$\mathcal{F}^{ch} = Cl^{ch}\mathbb{1} \quad \text{with}$$

$$\psi_\alpha(n)\mathbb{1} = 0, \quad \text{for } \alpha \in R_{>0} \text{ and } n \in \mathbb{Z}_{\geq 0}$$

$$\psi^\alpha(n)\mathbb{1} = 0, \quad \text{for } \alpha \in R_{>0} \text{ and } n \in \mathbb{Z}_{>0}.$$

and

$\deg(\mathbb{1}) = 0$, $\deg(\psi_\alpha(n)) = -1$, $\deg(\psi^\alpha(n)) = 1$.
for $\alpha \in R_{>0}$ and $n \in \mathbb{Z}$.

Associative case: Construction of $\mathcal{D}$

$$I_{>0,\chi} = \sum_{x \in \mathfrak{g}_{>0}} U(\mathfrak{g}_{>0})(x - \chi(x))$$

is a two sided ideal of $U(\mathfrak{g}_{>0})$.

$$\mathcal{D} = \frac{U(\mathfrak{g}_{>0})}{I_{>0,\chi}}.$$

Then $\mathcal{D}$ is a Heisenberg algebra of rank $\frac{1}{2}\dim(\mathfrak{g}_{\frac{1}{2}})$.

Associative case: Construction of Cl

$$Cl \cong \Lambda^\bullet(\mathfrak{g}_{>0}^*) \otimes \Lambda^\bullet(\mathfrak{g}_{>0})$$

is a vector space isomorphism where Cl is the Clifford algebra of $\mathfrak{g}_{>0} \oplus \mathfrak{g}_{>0}^*$ with respect to the bilinear form given by

$$(x_1 + \varphi_1 | x_2 + \varphi_2) = \varphi_1(x_2) + \varphi_2(x_1)$$

Then

$$\mathrm{gr}_K \mathcal{D} \cong \mathbb{C}[\chi + \nu(\mathfrak{g}_{-\frac{1}{2}})]$$

$$\mathrm{gr}_K Cl \cong \mathbb{C}[T^* \Pi \mathfrak{g}_{>0}^*]$$

as Poisson algebras.

Poisson case: Construction of $\bar{D}$

Let $\bar{I}_{>0,\chi}$ be the Poisson ideal of $\mathbb{C}[\mathfrak{g}^*_{>0}]$

generated by $\{x - \chi(x) \mid x \in \mathfrak{g}_{\geq 1}\}$.

$\chi + \nu(\mathfrak{g}_{-\frac{1}{2}})$ is an affine subspace of $\mathfrak{g}^*_{>0}$ and

$$\bar{D} = \frac{\mathbb{C}[\mathfrak{g}^*_{>0}]}{\bar{I}_{>0,\chi}} = \mathbb{C}[\chi + \nu(\mathfrak{g}_{-1/2})]$$

Poisson case: Construction of $\bar{Cl}$

$\Pi\mathfrak{g}^*_{>0}$ is $\mathfrak{g}^*_{>0}$ as a purely odd vectorspace

$T^*\Pi\mathfrak{g}^*_{>0}$ is the cotangent bundle

$$\bar{Cl} = \mathbb{C}[T^*\Pi\mathfrak{g}^*_{>0}] = \Lambda^\bullet(\mathfrak{g}^*_{>0} \oplus \mathfrak{g}_{>0}) = \Lambda^\bullet(\mathfrak{g}^*_{>0}) \otimes \Lambda^\bullet(\mathfrak{g}_{>0})$$

Then

$$\bar{C}(M) = M \otimes \bar{D} \otimes \bar{Cl}.$$

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Homework:

(1) Work out normal ordering and PBW.

(2) Virasoro normalization from Green-Scwarz-Witten

(3) Lie algebra cohomology and Fishel-Grojnowski-Teleman

(4) Work through §7 of Kac-Raan-Wakimoto.

(5) smooth $\hat{\mathfrak{g}}$-modules and restricted $\hat{\mathfrak{g}}$-modules.

A ~~$\hat{\mathfrak{g}}$-mo~~ smooth $\hat{\mathfrak{g}}$-module, or restricted $\hat{\mathfrak{g}}$-module, is a $\hat{\mathfrak{g}}$-module M such that

$$(xt^n)m = 0, \text{ for } x \in \mathfrak{g}, m \in M \text{ and } t \text{ large enough.}$$