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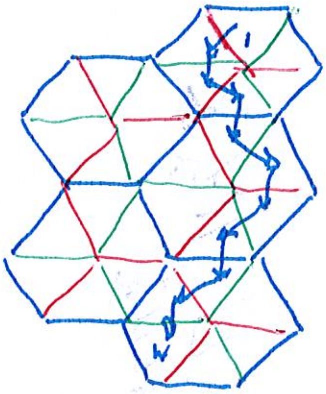
Combinatorics of the loop Grassmannian

14.06.2016

The loop Grassmannian



$$x_r(c_1)n_r^{-1}\cdots x_g(c_q)n_g^{-1}\,I$$

$$x_r(c)=\begin{pmatrix}1&c&0\\0&1&0\\0&0&1\end{pmatrix}\qquad x_g(c)=\begin{pmatrix}1&0&0\\0&1&c\\0&0&1\end{pmatrix}\qquad x_b(c)=\begin{pmatrix}1&0&0\\0&1&0\\ct&0&1\end{pmatrix}$$

$$n_r^{-1}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}\qquad n_g^{-1}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\qquad n_b^{-1}=\begin{pmatrix}0&0&-t^{-1}\\0&1&0\\t&0&0\end{pmatrix}$$

$$G=SL_3(\mathbb{C}[t,t^{-1}])\qquad K=SL_3(\mathbb{C}[t])$$

$$I=\left\{\begin{pmatrix}a_1&&b_{ij}\\&a_2&\\c_{ij}&&a_3\end{pmatrix}\in G\ \middle|\ \begin{array}{l}a_i\in\mathbb{C}[t]\quad a_i(0)\in\mathbb{C}^\times\\ b_{ij}\in\mathbb{C}[t]\\ c_{ij}\in t\mathbb{C}[t]\end{array}\right\}$$

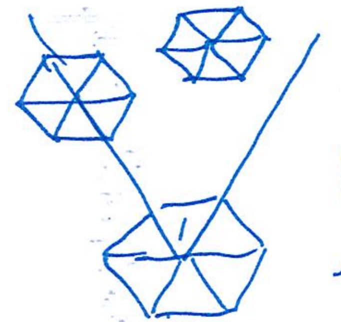
G/K is the <u>loop Grassmannian</u>

G/I is the <u>affine flag variety</u>

Decompositions

$W = \{\text{triangles}\} \;\supseteq\; \mathfrak{a}_{\mathbb{Z}} = \{\text{hexagons}\} \;\supseteq\; \mathfrak{a}_{\mathbb{Z}}^{+} = \left\{\begin{matrix}\text{hexagons}\\ \text{which}\\ \text{intersect}\\ \text{the } V \text{ cone}\end{matrix}\right\}$

$= \{$  $\}$

$= \{$  $\}$

$$G = \bigsqcup_{w \in W} I n_w I \qquad G = \bigsqcup_{\mu \in \mathfrak{a}_{\mathbb{Z}}} I t_\mu K. \qquad G = \bigsqcup_{\lambda \in \mathfrak{a}_{\mathbb{Z}}^{+}} K t_\lambda K$$

$$G = \bigsqcup_{v \in W} U^{-} v I \qquad G = \bigsqcup_{\mu \in \mathfrak{h}_{\mathbb{Z}}} U^{-} t_\mu K.$$

$$U^{-} = \left\{ \begin{pmatrix} 1 & & 0 \\ & 1 & \\ c_{ij} & & 1 \end{pmatrix} \;\middle|\; c_{ij} \in \mathbb{C}[t, t^{-1}] \right\}$$

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Sprouts: green folded paths

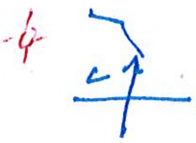
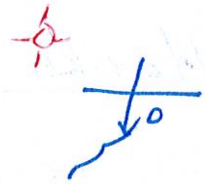
A green step of type j is

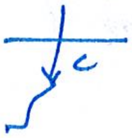
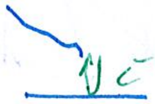
colour j  or  or  $c \neq 0$

Convert a blue labeled walk

to a green folded labeled path

Inductively, step by step,

 =  and  = 

 =  for $c \neq 0$.

Hence, decompose

$$Kt_\lambda K \cap U^- t_\mu K = \bigsqcup_{p \in B(\overrightarrow{w_0 t_\lambda})_{t_\mu}} C_p$$

where

$$B(\overrightarrow{w_0 t_\lambda})_{t_\mu} = \left\{ \begin{array}{l} \text{green folded paths } p \\ \text{of type } \overrightarrow{w t_\lambda} \\ \text{that end in } t_\mu \end{array} \right\}$$

$C_p = \{\text{labellings of } p\}$

Hecke algebras

The *affine Hecke algebra* H is

$$H = \mathbb{C}\text{span}\{IwI \mid w \in W\} \text{ with}$$

$$(IuI)(IvI) = \sum_{w \in W} h_{uv}^{w} (IwI)$$

where $h_{uv}^{w} = \#$ of IwI in $(IuI)(IvI)$

The *spherical Hecke algebra* H^{sph} is

$$H^{sph} = \mathbb{C}\text{-span}\{Kt_\lambda K \mid \lambda \in \mathfrak{a}_{\mathbb{Z}}^{+}\} \text{ with}$$

$$(Kt_\mu K)(Kt_\nu K) = \sum_{\lambda \in \mathfrak{a}_{\mathbb{Z}}^{+}} c_{\mu\nu}^{\lambda} \, Kt_\lambda K.$$

where $c_{\mu\nu}^{\lambda} = \#$ of $Kt_\lambda K$ in $(Kt_\mu K)(Kt_\nu K)$

The *polynomial algebra* is

$$\mathbb{C}[X] = \mathbb{C}\text{span}\{X^\mu \mid \mu \in \mathfrak{a}_{\mathbb{Z}}\} \text{ with } X^\mu X^\nu = X^{\mu+\nu}$$

Let $W_0 = \{$triangles in the 1-hexagon$\}$,

$$\mathbb{1}_0 = \sum_{w \in W_0} (IwI) \quad \text{and} \quad wX^\mu = X^{w\mu}$$

for $\mu \in \mathfrak{a}_{\mathbb{Z}}$, $w \in W_0$.

Weyl characters s_λ

Jacobi:

$$\mathbb{C}[X]^{W_0} \xrightarrow{\sim} \mathbb{C}[X]^{\det}$$
$$f \longmapsto a_\rho f$$
$$s_\lambda \longleftarrow a_{\lambda+\rho} = \sum_{w \in W_0} \det(w)\, w X^{\lambda+\rho}$$

$\mathbb{C}[X]^{\det} = \{ f \in \mathbb{C}[X] \mid wf = \det(w) f \text{ for } w \in W_0 \}$

is a free $\mathbb{C}[X]^{W_0}$-module with

generator $\quad a_\rho =$ Vandermonde determinant

$\qquad\qquad\qquad =$ Weyl denominator

Hermann Weyl: $\quad G^\vee = PGL_3(\mathbb{C})$

$L(\lambda)$ the finite dim'l $G^\vee$-module,

simple of highest weight λ.

$$s_\lambda = \mathrm{char}(L(\lambda)).$$

Satake-Casselman-Shalika-Macdonald-Bernstein-Lusztig

$$\mathbb{C}[X]^{W_0} \xrightarrow{\sim} H^{sph} = \mathbb{1}_0 H \mathbb{1}_0$$

spherical function $\quad P_\lambda(0, q^{-1}) \longleftarrow K t_\lambda K \quad$ double coset

$s_\lambda \longmapsto C_\lambda \quad$ Kazhdan-Lusztig basis element.

Crystals and MV-cycles

$L(\lambda)$ has a special basis $\left\{ v_p \;\middle|\; \begin{array}{c} p \text{ is a} \\ \text{Littelmann} \\ \text{path} \end{array} \right\}$

The *crystal of highest weight λ* is

$$B(\lambda) = \{ p \mid p \text{ is a Littelmann path} \}$$

endowed with root operators. Recall

$$Kt_\lambda K \cap Kt_\mu K = \bigcup_{p \in B(\overrightarrow{w_0 t_\lambda})_{t_\mu}} C_p$$

Define

$$\dim(p) = \dim(C_p) = (\#\text{ of } \uparrow \text{ in } p) + (\#\text{ of } \circlearrowleft \text{ in } p)$$

A *Littelmann path* is

$$p \in B(\overrightarrow{w_0 t_\lambda})_{t_\mu} \quad \text{with } \dim(p) \text{ maximal.}$$

A *Mirković–Vilonen cycle* is an element of

$$\left\{ \begin{array}{c} \text{irreducible} \\ \text{components} \\ \text{of} \\ \hline Kt_\lambda K \cap Ut_\mu K \end{array} \right\} = \{ C_p \mid p \text{ is a Littelmann path} \}$$