Combinatorics of representations of affine Lie algebras A. Ram

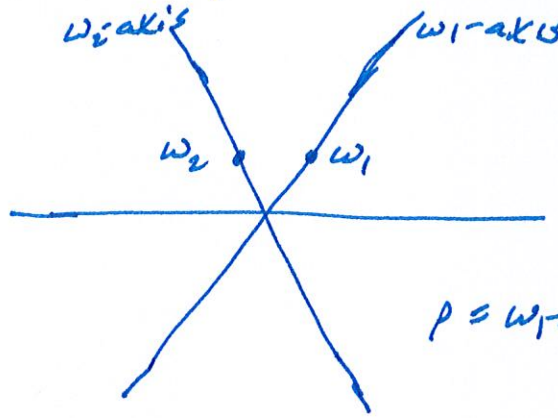
Verma modules

Univ. of Sydney 16.06.2016

Example: $\mathfrak{g} = \mathfrak{sl}_3 = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{n}^- \oplus \mathfrak{b}$

Fin. dim. simple $\mathfrak{h}$-modules $\mathbb{C}_\lambda$ are indexed by

$\lambda \in \mathfrak{h}^* = \mathbb{C}\omega_1 + \mathbb{C}\omega_2$



The Verma module

$$M(\lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$$

where $\mathbb{C}_\lambda = \mathbb{C}v_\lambda$ with $hv_\lambda = \lambda(h)v_\lambda$ and $xv_\lambda = 0$ for $h \in \mathfrak{h}$, $x \in \mathfrak{n}^+$.

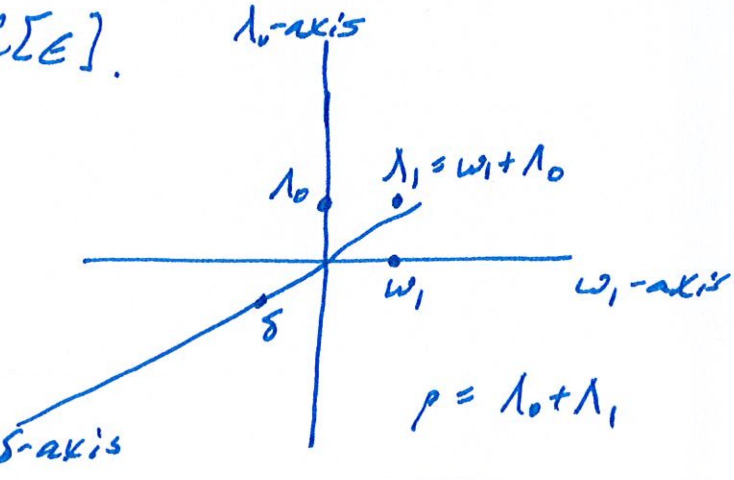
Example $\mathfrak{g} = \widehat{\mathfrak{sl}}_2 = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{n}^+ \oplus \mathfrak{b}$

$$\mathfrak{n}^- = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \,\middle|\, \begin{matrix} a_{11} + a_{12} = 0 \\ a_{ij} \in \varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}] \end{matrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{C} \right\}$$

$$\mathfrak{h} = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}K \oplus \mathbb{C}d = \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}K \oplus \mathbb{C}d$$

$$\mathfrak{n}^+ = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathfrak{sl}_2 \otimes_{\mathbb{C}} \varepsilon\mathbb{C}[\varepsilon].$$

$$\mathfrak{h}^* = \mathbb{C}\omega_1 \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$$



The Weyl group of λ

Theorem There is a set $(R^\vee)^+$ such that

$M(\lambda)$ is simple if and only if $\lambda \notin \bigcup_{\substack{\alpha\in(R^\vee)^+ \\ k\in\mathbb{Z}_{>0}}} \mathfrak{h}^{\alpha,k}$

where $\mathfrak{h}^{\alpha,k} = \{\mu\in\mathfrak{h}^* \mid \langle\mu+\rho,\alpha^\vee\rangle = k\}$.

$$L(\lambda) = \frac{M(\lambda)}{\langle\text{max. proper submodule}\rangle}$$

The Weyl group of λ is

$$W = \langle s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^* \mid \alpha\in(R^\vee)^+ \text{ and } \langle\lambda+\rho,\alpha^\vee\rangle\in\mathbb{Z}\rangle$$

where $s_\alpha\circ\mu = \mu - \langle\mu+\rho,\alpha^\vee\rangle\alpha$. Let

ν be the "antidominant" rep of $W\circ\lambda$

$W\circ\nu = W\circ\lambda = \{w\circ\nu \mid w\in W^\nu\}$

$W_\nu = \mathrm{Stab}_W(\nu)$

Linkage Let $w\circ\nu\in W\circ\nu$. All composition factors

of $M(w\circ\nu)$ are in $\{L(v\circ\nu) \mid v\in W^\nu\}$

The Hecke algebra H of λ

H is the $\mathbb{Z}[t^{\frac12},t^{-\frac12}]$-algebra generated by

$T_1,\dots,T_r$ with $T_i^2=(t^{\frac12}-t^{-\frac12})T_i+1$

$$\underbrace{T_iT_jT_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}\text{ factors}}$$

if W has Coxeter presentation with

$s_1,\dots,s_r$ and $s_i^2=1$ and $(s_is_j)^{m_{ij}}=1$.

Let $\mathbf{1}_\nu$ be a symbol such that

$$T_i\mathbf{1}_\nu = t^{\frac12}\mathbf{1}_\nu \quad \text{if} \quad s_i\in W_\nu.$$

The <u>bar</u> is $\mathbb{Z}$-linear $\bar{\ }: H\to H$ and $\bar{\ }: H\mathbf{1}_\nu\to H\mathbf{1}_\nu$

$$\overline{t^{\frac12}}=t^{-\frac12},\quad \overline{T_i}=T_i^{-1},\quad \overline{\mathbf{1}_\nu}=\mathbf{1}_\nu,\quad \overline{h_1h_2\mathbf{1}_\nu}=\overline{h_1}\,\overline{h_2\mathbf{1}_\nu}$$

$H\mathbf{1}_\nu$ has

standard basis $\{T_w\mathbf{1}_\nu \mid w\in W^\nu\}$

where $T_w=T_{i_1}\cdots T_{i_\ell}$ if $w=s_{i_1}\cdots s_{i_\ell}$ is min. length.

KL-basis $\{C_w\mathbf{1}_\nu\mid w\in W^\nu\}$ with

$$\overline{C_w\mathbf{1}_\nu}=C_w\mathbf{1}_\nu$$

$$C_w\mathbf{1}_\nu = T_w\mathbf{1}_\nu+\sum_{v\ne w} p_{vw}T_v\mathbf{1}_\nu \quad \text{with } p_{vw}\in t^{\frac12}\mathbb{Z}[t^{\frac12}]$$

The Jantzen conjecture

$$M(\lambda) = M(\lambda)^{(1)} \supseteq M(\lambda)^{(2)} \supseteq \ldots$$

is the Jantzen filtration of $M(\lambda)$

$$K(\mathcal{O}[\nu]) = \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]\text{-span}\{S_{w\circ\nu} \mid w \in W^\nu\}$$

Bases:

$$\{S_{w\circ\nu} \mid w \in W^\nu\} \text{ and } \{V_{w\circ\nu} \mid w \in W^\nu\}$$

with

$$V_{w\circ\nu} = \sum_{v \in W^\nu} \left(\sum_j t^j \left[\frac{M(w\circ\nu)^{(j)}}{M(w\circ\nu)^{(j+1)}} : L(v\circ\nu) \right] \right) S_{v\circ\nu}$$

Then

$$K(\mathcal{O}[\nu]) \xrightarrow{\sim} H\mathbb{1}_\nu$$

$$V_{w\circ\nu} \longmapsto T_w \mathbb{1}_\nu$$

$$S_{v\circ\nu} \longmapsto C_v \mathbb{1}_\nu$$

Weyl modules

$$\mathfrak{g} = \mathfrak{n}_0^- \oplus \mathfrak{g}_0 \oplus \mathfrak{n}_0^+ = \mathfrak{n}_0^- \oplus \mathfrak{b}_0$$

P_0^+ an index set for fin. dim'l $\mathfrak{g}_0$-modules. $L_{\mathfrak{g}_0}(\lambda)$

The Weyl module

$$\Delta(\lambda) = \mathrm{Ind}_{\mathfrak{b}_0}^{\mathfrak{g}}(L_{\mathfrak{g}_0}(\lambda)) = U\mathfrak{g} \otimes_{U\mathfrak{b}_0} L_{\mathfrak{g}_0}(\lambda)$$

with $xm = 0$ for $x \in \mathfrak{n}_0^+$ and $m \in L_{\mathfrak{g}_0}(\lambda)$

Example $\mathfrak{g} = \hat{\mathfrak{sl}}_2$

$$= \mathfrak{sl}_2 \otimes \epsilon^{-1}\mathbb{C}[\epsilon^{-1}] \oplus (\mathfrak{sl}_2 \oplus \mathbb{C}K + \mathbb{C}d) \oplus \mathfrak{sl}_2 \otimes \epsilon\mathbb{C}[\epsilon]$$

H_0 is the subalgebra of H corresponding to Weyl group of λ in $\mathfrak{g}_0$

e_0 is a symbol with $e_0 T_i = -t^{-\frac{1}{2}} e_0$ for $T_i \in H_0$

Using a Jantzen filtration of $\Delta(\lambda)$ gives

$$K(\mathcal{O}_{\mathfrak{g}_0}^{\mathfrak{g}}[\nu]) \longrightarrow e_0 H \mathbf{1}_\nu$$

$$S_{w_0 w \circ \nu} \longmapsto e_0 T_w \mathbf{1}_\nu$$

$$V_{w_0 w \circ \nu} \longmapsto C_{w_0 w} \mathbf{1}_\nu$$