Combinatorics of affine Springer fibers

Parking functions

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A *parking function* is $f:\{1,\ldots,n\}\to\mathbb{Z}_{\geq 0}$ such that $(f(1),\ldots,f(n))$ arranged in increasing order $(a_1,\ldots,a_n)$ satisfies $a_i \leq i-1$

A generalization: $d = mn+b$

A $\frac{d}{n}$ *parking function* is $f:\{1,\ldots,n\}\to\mathbb{Z}_{\geq 0}$ such that $(f(1),\ldots,f(n))$ arranged in increasing order $(a_1,\ldots,a_n)$ satisfies $a_i \leq mi+b$

Examples $n=2$: $(0,0)$ $(0,1)$ $(1,0)$ 3 total

$n=3$: $(0,0,0)$ $(0,0,1)$ $(0,1,0)$ $(1,0,0)$ $(0,1,1)$ $(1,0,1)$ $(1,1,0)$ $(0,0,2)$ $(0,2,0)$ $(2,0,0)$ $(0,1,2)$ $(0,2,1)$ $(1,0,2)$ $(2,0,1)$ $(1,2,0)$ $(2,1,0)$

16 total

Arrangements

Type GL_n For $1 \le i < j \le n$

$$H^{\varepsilon_i - \varepsilon_j} = \{(x_1, \ldots x_n) \in \mathbb{R}^n \mid x_i - x_j = 0\}$$

$$\mathcal{A}^0 = \{H^{\varepsilon_i - \varepsilon_j} \mid 1 \le i < j \le n\}$$

$$H^{-(\varepsilon_i - \varepsilon_j) + k\delta} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = k\}.$$

$$\mathcal{A}^k = \{H^{-(\varepsilon_i - \varepsilon_j) + k\delta} \mid 1 \le i < j \le n\}.$$

Type G For $\alpha \in R^+$

$$H^\alpha = \{x \in \mathfrak{a}_\mathbb{R} \mid \langle x, \alpha \rangle = 0\}$$

$$\mathcal{A}^0 = \{H^\alpha \mid \alpha \in R^+\}.$$

$$H^{-\alpha + k\delta} = \{x \in \mathfrak{a}_\mathbb{R} \mid \langle x, \alpha \rangle = k\}.$$

$$\mathcal{A}^k = \{H^{-\alpha + k\delta} \mid \alpha \in R^+\}.$$

The braid arrangement is $\mathcal{A}^0$

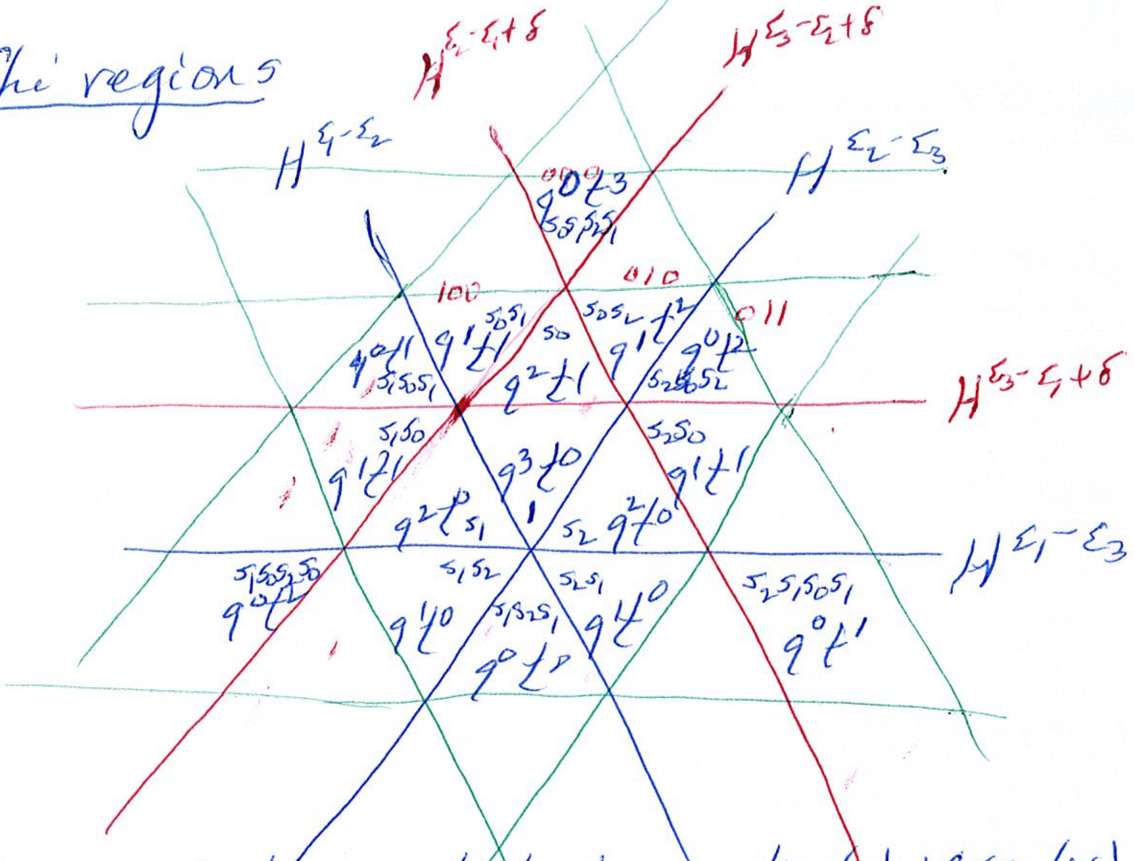
The finite Weyl group is $W_0 = \{\text{conn. comp. of } \mathfrak{a}_\mathbb{R} \setminus \mathcal{A}^0\}$

The affine arrangement is $\mathcal{A}^\mathbb{Z} = \bigcup_{k \in \mathbb{Z}} \mathcal{A}^k$.

The affine Weyl group is $W_\mathbb{Z} = \{\text{conn. comp. of } \mathfrak{a}_\mathbb{R} \setminus \mathcal{A}^\mathbb{Z}\}$

The Shi arrangement is $\mathcal{A}^0 \cup \mathcal{A}^1$

Shi regions

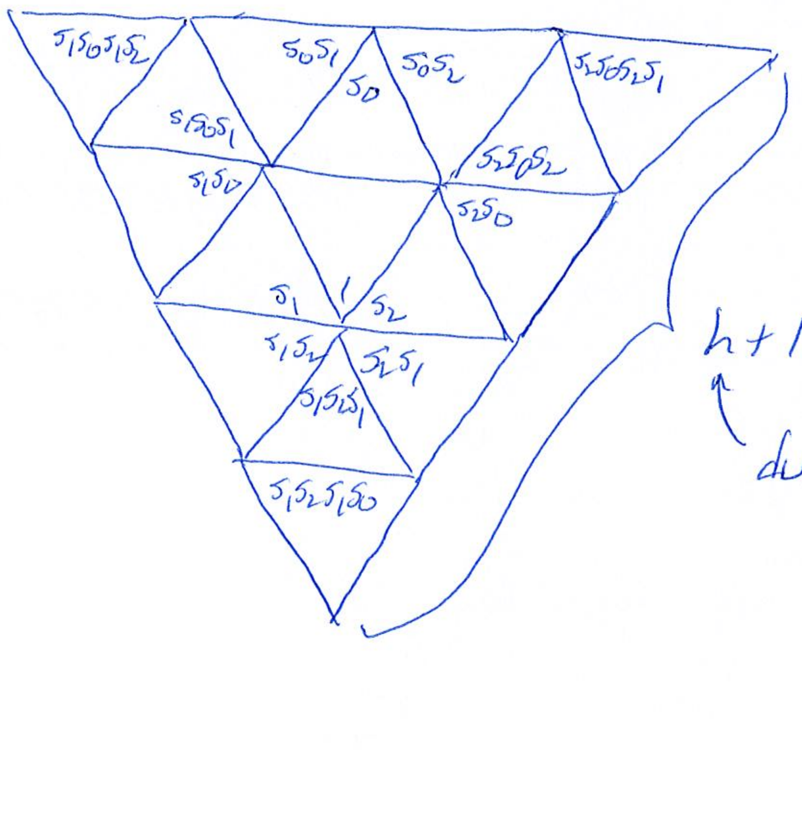


Area and dinv statistics $q^{dinv(w)}t^{area(w)}$

$q^{dinv(w)} = q^{\#\text{blue-red between } w_0 \text{ and } w}$, $t^{area(w)} = t^{diag(w)}$ if w dominant

There is a bijection $W_{\mathbb{Z}} \longrightarrow W_{\mathbb{Z}}$

$w \longmapsto w^{-1}$



$h+1$

dual Coxeter number

Representations of DAHA

The DAHA is given by generators

$t_0, t_1, \dots, t_n$ and $Y^{\varepsilon_1}, Y^{\varepsilon_2}, \dots, Y^{\varepsilon_n}$

with relations

The nonsymmetric Macdonald polynomials $\{E_w \mid w \in W_1\}$ form a basis of $L_{1+\frac{1}{n}}(\mathrm{triv})$ an irreducible finite dimensional DAHA module.

$$\begin{tikzcd}[column sep=small]
 & & & E_{000} \arrow[d, "t_0"] & & & \\
 & & & E_{210} \arrow[dl, "t_1"'] \arrow[dr, "t_2"] & & & \\
 & & E_{120} \arrow[d, "t_2"'] & & E_{201} \arrow[d, "t_1"] & & \\
 & & E_{102} \arrow[dl, "t_0"'] \arrow[dr, "t_1"] & & E_{021} \arrow[dl, "t_2"'] \arrow[dr, "t_0"] & & \\
 & E_{300} \arrow[d, "t_1"] & & E_{012} \arrow[d, "t_0"] & & E_{330} \arrow[d, "t_2"] & \\
 & E_{030} \arrow[dl, "t_2"'] \arrow[dr, "t_0"] & & E_{420} \arrow[dl, "t_1"'] \arrow[dr, "t_2"] & & E_{303} \arrow[dl, "t_0"'] \arrow[dr, "t_1"] & \\
E_{003} & & E_{240} & & E_{402} & & E_{033}
\end{tikzcd}$$

$$Y^{\varepsilon_1} E_{420} = q^4 E_{420}$$
$$Y^{\varepsilon_2} E_{420} = q^2 E_{420}$$
$$Y^{\varepsilon_3} E_{420} = q^0 E_{420}$$

Affine Springer fibers

$G = GL_n(\mathbb{C}[t,t^{-1}])$

$$I = \left\{ \begin{pmatrix} a_1 & & b_{ij} \\ & \ddots & \\ c_{ij} & & a_n \end{pmatrix} \;\middle|\; \begin{array}{l} a_i \in \mathbb{C}[t],\ a_i(0) \in \mathbb{C}^\times \\ b_{ij} \in \mathbb{C}[t] \\ c_{ij} \in t\mathbb{C}[t] \end{array} \right\}$$

G/I is the affine flag variety.

G acts on $\mathfrak{g} = M_n(\mathbb{C}[t,t^{-1}])$ by conjugation.

$$v = t^m \begin{pmatrix} 0 & \overbrace{\cdots}^{b} & 1 & & \\ & \ddots & & \ddots & \\ & & \ddots & & 1 \\ t & & & \ddots & \\ & \underbrace{\ddots}_{b} & t & & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{b} = \left\{ \begin{pmatrix} a_1 & & b_{ij} \\ & \ddots & \\ c_{ij} & & a_n \end{pmatrix} \;\middle|\; \begin{array}{l} a_i \in \mathbb{C}[t] \\ b_{ij} \in \mathbb{C}[t] \\ c_{ij} \in t\mathbb{C}[t] \end{array} \right\}$$

The affine Springer fiber is

$$\mathcal{B}_{\frac{mn+b}{n}} = \{ gI \in G/I \mid gvg^{-1} \in \mathfrak{b} \}.$$

Then

$H^*_{\mathbb{G}_m}(\mathcal{B}_{\frac{mn+b}{n}})$ is a fin. dim'l DAHA module,

$L_{m+\frac{b}{n}}(\mathrm{triv})$

Oblomkov-Yun explain

$\mathrm{gr}_* H^*_{\mathbb{G}_m}(\mathcal{B}_{\frac{mn+b}{n}})$ is a rational Cherednik algebra module.

By restriction

$$gr_* H^*_{\mathbb{G}_m}(B_{\frac{mn+b}{n}}) \text{ is an } S_n\text{-module}.$$

Then, the generalized shuffle conjecture at $b=1$,

$$\sum_{\lambda\vdash n} \sum_{i,j} t^i q^j \, \underset{\text{char}}{\overset{\text{Frobenius}}{\cancel{\dim}}} \left(gr_i H^j\left(B_{\frac{mn+b}{n}}\right)\right) = Q_{m,n} 1\!\!1.$$

Hikita explained that the monomial expansion of the LHS is a direct consequence of Goresky-Kottwitz-MacPherson:

$$B_{\frac{mn+b}{n}} \cap I w I \cong \mathbb{C}^{\dim v(w)}$$

where $\dim v(w) = \#$ of blue $- \#$ red between w_0 and w.