A. Ram Fusion talk
05.10.2016
Working seminar
Univ. of Melbourne

Three definitions of fusion

$$T(W) = \widehat{W}(-\infty)$$

$$\dot{W} = \varinjlim_{r} \varprojlim_{t} \overline{W_{t,r}}$$

$$T'(W) = \mathrm{Res}^{\hat{\mathfrak{g}}^{\heartsuit}}_{\check{\mathfrak{g}}^{\heartsuit}}(Z^{\infty})$$

Relations: $T(W) = \dot{W}$ and $T(W)^{\#} = \mathcal{D}(T'(W))$

Lie algebras:

$$\hat{\mathfrak{g}}^{\heartsuit} = \mathbb{C}((\epsilon))^{\heartsuit} \otimes \mathfrak{g} \oplus \mathbb{C}\mathbf{1}$$

$$\check{\mathfrak{g}}^{\heartsuit} = \mathbb{C}[\epsilon, \epsilon^{-1}]^{\heartsuit}\ \cancel{\mathbb{C}[[\epsilon]]^{\heartsuit}} \otimes \mathfrak{g} \oplus \mathbb{C}\mathbf{1}$$

$$\Gamma = R \otimes \mathfrak{g} \oplus \mathbb{C}\mathbf{1}.$$

$R = \{\text{regular functions } (C - \{p_s \mid s \in S\}) \xrightarrow{f} \mathbb{C}\}$

C is a smooth projective curve

S an index set for marked points p_s on C

$\heartsuit \subseteq S$ indexing connected components of C.

If $s_0 \in \heartsuit$ then $p_s \in C_{s_0}$ for $s \in [s_0]$

(Assume $\mathrm{Card}([s_0]) \geq 2$).

A. Ram Fusion talk
05.10.2016
Working seminar
Univ. of Melbourne

If $\heartsuit = \{1, 2, \dots, k\}$ then

$$\tilde{\mathfrak{g}}^{\heartsuit} = \mathbb{C}[\epsilon_1, \epsilon_1^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}[\epsilon_2, \epsilon_2^{-1}] \otimes \mathfrak{g} \oplus \cdots \oplus \mathbb{C}[\epsilon_k, \epsilon_k^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}\mathbf{1}.$$

In Shimizu-Ueno ~~Definition 4.21~~,

$$\hat{\mathfrak{g}}_N = \bigoplus_{j=1}^{N} \mathfrak{g} \otimes \mathbb{C}((\xi_j)) \oplus \mathbb{C}c \qquad \text{Definition 4.21}$$

$$Gr_\bullet(\hat{\mathfrak{g}}_N) = \mathfrak{g} \otimes_{\mathbb{C}} \Big(\bigoplus_{j=1}^{N} \mathbb{C}[\xi_j; \xi_j^{-1}] \Big) \oplus \mathbb{C}c \qquad \left(\begin{matrix}\text{just before} \\ (4.38)\end{matrix}\right)$$

$$\hat{\mathfrak{g}}(\mathfrak{X}) = \mathfrak{g} \otimes H^0\Big(C, \mathcal{O}_C\big(* \textstyle\sum_{j=1}^{N} Q_j\big)\Big)$$

where $\mathfrak{X} = (C; Q_1, \dots, Q_N)$ is a curve with marked points.

<u>The module W</u>: Let

$$\spadesuit = S - \heartsuit$$

$\{V_s \mid s \in \spadesuit\}$ a collection of smooth $\hat{\mathfrak{g}}$ modules of level $\kappa - h$.

$$W = \bigotimes_{s \in \spadesuit} V_s$$

In Shimizu-Ueno,

$$\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_N} \qquad \text{(right after Lemma 4.22).}$$

A. Ram Fusion talk
05.10.2016
Working seminar
Univ. of Melbourne

The projective limit $\hat{W}$

$$R_1 = \{ f \in R \mid {}^{s_0}f \in \mathbb{C}[[\varepsilon]] \text{ for } s_0 \in S^1 \}$$

$$G_N = \mathrm{span}\left\{ (f_1c_1)(f_2c_2)\cdots(f_Nc_N) \;\middle|\; \begin{matrix} f_1,\ldots,f_N \in R_1 \\ c_1,\ldots,c_N \in \mathfrak{g} \end{matrix} \right\}$$

so that $G_N \subseteq U(\Gamma)$. Then

$$W \supseteq G_1 W \supseteq G_2 W \supseteq \cdots$$

gives $\quad W/G_1W \leftarrow W/G_2W \leftarrow \cdots$

and $$\hat{W} = \varprojlim_k \left(\frac{W}{G_kW} \right)$$

(W is an R-module by restriction only to the points in S).

Smooth vectors

The smooth vectors in V are the elements of $V(\infty)$.

Fact: $\quad V(\infty) = V^{\#}(-\infty)$

Here

$$V(\infty) = \bigcup_{N \in \mathbb{Z}_{\geq 0}} V(N) \quad \text{and} \quad V(-\infty) = \bigcup_{N \in \mathbb{Z}_{\leq 0}} V(N)$$

where

$$V(0) \subseteq V(1) \subseteq V(2) \subseteq \cdots$$

$$V(0) \subseteq V(-1) \subseteq V(-2) \subseteq \cdots$$

and

$$V(N) = \{x \in V \mid Q_N x = 0\} \quad \text{for } N \in \mathbb{Z}_{\geq 0}$$

$$V(-N) = \{x \in V \mid Q_N^{\#} x = 0\} \quad \text{for } N \in \mathbb{Z}_{\geq 0}.$$

and

$$Q_N = \mathrm{span}\{(\varepsilon c_1)\cdots(\varepsilon c_N) \mid c_1, \ldots, c_N \in \mathfrak{g}\}$$

is a subspace of $U(\hat{\mathfrak{g}})$

<u>The Lie algebra involution</u>

$$\begin{aligned} \#: \hat{\mathfrak{g}} &\to \hat{\mathfrak{g}} \\ \varepsilon^n c &\mapsto -\varepsilon^{-n} c \\ \mathbb{1} &\mapsto -\mathbb{1}. \end{aligned}$$

So $T(W)$ is the "smooth part" of the $\hat{\mathfrak{g}}^{\mathcal{Q}}$-module $\hat{W}$.

A. Ram Fusion talk
05.10.2016
Working seminar
Univ. Melbourne.

Defining $\dot{W}$

$$W_{t,r} = \{x \in W \mid \text{if } \xi \in H_r \text{ then } \xi x \in G_{t-r}W\}$$

$$\overline{W_{t,r}} = \frac{W_{t,r}}{G_t W}$$

$$\dot{W}_r = \prod_{t \geq r} \overline{W_{t,r}} \quad \text{and} \quad \dot{W} = \bigcup_{r \in \mathbb{Z}_{\geq 1}} \dot{W}_r$$

(here $\dot{W}_1 \subseteq \dot{W}_2 \subseteq \cdots$).

The subspace H_r

$$H_r = \mathrm{span}\left\{ (f_{s_1} c_1) \cdots (f_{s_r} c_r) \;\middle|\; \begin{matrix} c_1, \ldots, c_r \in \mathfrak{g} \\ s_1, \ldots, s_r \in \heartsuit \end{matrix} \right\}$$

and f_s is regular on $C - \{p_s\}$ and has ${}^s f_s = \frac{1}{\epsilon}$

So, I think

$$H_r = \mathrm{span}\left\{ (\epsilon_{s_1}^{-1} c_1) \cdots (\epsilon_{s_r}^{-1} c_r) \;\middle|\; \begin{matrix} c_1, \ldots, c_r \in \mathfrak{g} \\ s_1, \ldots, s_r \in \heartsuit \end{matrix} \right\}$$

One point to make is that

$$H_r G_t \subseteq G_{t-r} U(\Gamma).$$

Defining $T'(W)$

$$Z = \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$$

$$Z^N = \mathrm{Ann}_Z(G_N W) \quad \text{and} \quad Z^\infty = \bigcup_{N \in \mathbb{Z}_{\geq 0}} Z^N.$$

Then

$$T'(W) = \mathrm{Res}^{\hat{g}^\varphi}_{\check{g}^\varphi}(Z^\infty).$$