Working seminar
R-matrices and quantum groups
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①

Ingredients: E is an elliptic curve.

$A = U(1)^n$ = diag matrices in $U(n)$, $T = A \times U(1) = U(1)^{n+1}$

$Gr(k,n) = \{k\text{ dim subsp of } \mathbb{C}^n\}$ $X_{k,n} = T^*(Gr_{k,n})$

$X_n = \bigsqcup_{k=0}^{n} X_{k,n}$, $\hat{E}_T(X_n) = \bigsqcup_{k=0}^{n} \hat{E}_T(X_{k,n})$ $H_G^{ell}(X_n)_{\mathcal{L}} = \bigoplus_{k=0}^{n} H_G^{ell}(X_{k,n})_{\mathcal{L}}$.

Base spaces

$E_T(X_{k,n}) = E_A(Gr(k,n)) \times E$, $\hat{E}_T(X_{k,n}) = E_T(X_{k,n}) \times E$

$E_G(X_{k,n}) = E_{U(n)}(Gr(k,n)) \times E$, $\hat{E}_G(X_{k,n}) = E_G(X_{k,n}) \times E$

Sheaves $\mathcal{J}_{k,n} = c^*\mathcal{L}(N_{k,n}, 0)$, $p_T : \hat{E}_T(X_{k,n}) \to \hat{E}_T(pt)$

$c : \hat{E}_T(X_{k,n}) \xrightarrow{\chi \times id \times id} E^{(k)} \times E^{(n-k)} \times E \times E$

$\mathcal{H}_T^{ell}(X_{k,n}) = (p_T)_* \mathcal{J}_{k,n}$ on $\hat{E}_T(pt)$

$\mathcal{H}_G^{ell}(X_{k,n}) = \pi_* \mathcal{H}_T^{ell}(X_{k,n})^{S_n}$ on $\hat{E}_G(pt)$

Sections:

$H_T^{ell}(X_{k,n})_{\mathcal{L}} = \Gamma(\hat{E}_T(pt), \mathcal{H}_T^{ell}(X_{k,n}) \otimes \mathcal{L})$

$H_G^{ell}(X_{k,n})_{\mathcal{L}} = \Gamma(\hat{E}_G(pt), \mathcal{H}_G^{ell}(X_{k,n}) \otimes \mathcal{L})$

Coordinates:

$E_T(pt) = E^n \times E$ and $E_G(pt) = E^{(n)} \times E$.

The elliptic dynamical R-matrix for $\mathfrak{gl}_n$ is

$$R(z,y,\lambda) = \sum_{i=1}^{n} E_{ii}\otimes E_{ii} + \sum_{i\neq j} \alpha(z,y,\lambda_i-\lambda_j) E_{ii}\otimes E_{jj} + \sum_{i\neq j} \beta(z,y,\lambda_i-\lambda_j) E_{ij}\otimes E_{ji}.$$

where $\alpha(z,y,\lambda) = \dfrac{\theta(z)\theta(\lambda+y)}{\theta(z-y)\theta(\lambda)}$ $\quad \beta(z,y,\lambda) = \dfrac{\theta(z+\lambda)\theta(y)}{\theta(z-y)\theta(\lambda)}$

Here $R: \mathbb{C}\times\mathbb{C}\times\mathfrak{h}^* \longrightarrow \mathrm{End}_{\mathfrak{h}}(V\otimes V)$ and the

$$(z, y, \lambda) \longmapsto R(z,y,\lambda)$$

spectral parameter (z), deformation parameter (y), dynamical parameter (λ)

dynamical Yang-Baxter equation is

$$R(z,y,\lambda-yh^{(3)})^{(12)} R(z+w,y,\lambda)^{(13)} R(w,y,\lambda-yh^{(1)})^{(23)}$$
$$= R(w,y,\lambda)^{(23)} R(z+w,y,\lambda-yh^{(2)})^{(13)} R(z,y,\lambda-yh^{(3)})^{(12)}$$

and the inversion relation is

$$R(z,y,\lambda)^{(12)} R(-z,y,\lambda)^{(21)} = \mathrm{Id}.$$

(if $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$ then $R(z,y,\lambda-yh^{(3)})^{(12)}$ acts as

$R(z,y,\lambda-y\mu_3)\otimes \mathrm{Id}$ on $V_{\mu_1}\otimes V_{\mu_2}\otimes V_{\mu_3}$.)

Appendix: Axiomatic definition of elliptic stable envelopes d'après Maulik-Okounkov.

Theorem A.1 $c^* w_I^+$ is a meromorphic section of the admissible line bundle $\rho_I^* \mathcal{L}(N, D) \otimes T_{k,n}$ with

$$\downarrow$$
$$\hat{E}_T(X_{k,n})$$

(a) Triangularity: $c^* w_I^+$ restricted to Y_J is 0 unless $J \le I$

(b) support condition $c^* w_I^+$ restricted to Y_J, as a function $\mathbb{C}^{n+2} \to \mathbb{C}$ is $\frac{1}{\Phi_I} \prod_{a \in J} \prod_{\substack{b \in \bar{J} \\ b < a}} \theta(z_a - z_b + y) F_{I,J}$

where F_{IJ} is holomorphic.

(c) Top term $c^* w_I^+$ restricted to Y_I, as a function $\mathbb{C}^{n+2} \to \mathbb{C}$ is

$$\frac{1}{\Phi_I} \frac{\prod_{a \in I} \prod_{b \in \bar{I}} \theta(z_a - z_b + \ell(a,b) y)}{\prod_{a \in I} \theta(\lambda - (w(a, I) + 1) y)}$$

§5.5 The stable envelope is the map

$$\mathrm{stab}: (\mathbb{C}^2)^{\otimes n} \longrightarrow \bigoplus_{k=0}^{n} \bigoplus_{\substack{I \subseteq \{1,\dots,n\} \\ |I| = k}} H_T^{ell}(X_{k,n})_{\mathcal{L}_I(D_I)}$$

$$v_I \longmapsto c^* w_I^+$$

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Elliptic Dynamical quantum $\mathfrak{gl}_2$ action

$L_{11}(w)$, $L_{12}(w)$, $L_{21}(w)$, $L_{22}(w)$ are generators of the elliptic dynamical quantum group $U^{ell}\mathfrak{gl}_2$

There is an action (see §6) on ~~sections~~ sections of an adm. line bundle on $\hat{E}_T(X_{k,n})$

The operators satisfy the elliptic dynamical quantum group relations

$$\mu_L R(w_1-w_2,y,\lambda)^{(12)} L(w_1)^{(13)} L(w_2)^{(23)}$$

$$= L(w_2)^{(23)} L(w_1)^{(13)} \mu_r R(w_1-w_2,y,\lambda)^{(12)}$$

Parallel to the case of Yangians and affine quantum enveloping algebras.

Gelfand-Zetlin subalgebra

The <u>Gelfand-Zetlin subalgebra</u> is generated by $L_n(w)$ and the determinant $\Delta(w)$.

Eigenbasis in $V(z_1) \otimes \cdots \otimes V(z_n)$ (tensor product of evaluation representations)

$$\{ \xi_I \mid I \subseteq \{1, 2, \ldots, n\} \}$$

Eigenvectors

$$\hat{\xi}_I = \sum_{|J|=k} \frac{W_J(z_I, z_J, y, \lambda)}{\prod_{\substack{a \in J \\ b \in I}} \theta(z_a - z_b + y)} \, \mathrm{Stab}(v_J)$$

The twisting line bundle $\mathcal{T}_{k,n}$

Let

$$p_T : \hat{E}_T(X_{k,n}) \longrightarrow \hat{E}_T(pt)$$

$$\chi : E_A(Gr(k,n)) \longrightarrow E_{U(k)}(pt) \times E_{U(n-k)}(pt) = E^{(k)} \times E^{(n-k)}$$

$$c : \hat{E}_T(X_{k,n}) \xrightarrow{\chi \times id \times id} E^{(k)} \times E^{(n-k)} \times E \times E$$

Then

$$\mathcal{T}_{n,k} = c^* \mathcal{L}(N_{k,n}, 0),$$

where $N_{k,n}$ is the quadratic form defined by

$$N_{k,n}(t_1,\ldots,t_k,s_1,\ldots,s_{n-k},y,\lambda) = 2\sum_{i=1}^{k} t_i(\lambda+(n-k)y) + \sum_{i=1}^{k}\sum_{j=1}^{n-k}(t_i-s_j)^2$$

Line bundles on E^p (Appel-Humbert).

$$E = \mathbb{C}/\Lambda = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \quad \text{and} \quad E^p = \mathbb{C}^p/\Lambda^p = \frac{\mathbb{C}^p}{(\mathbb{Z}+\tau\mathbb{Z})^p}$$

N is a symmetric matrix, the matrix of a bilinear form.

$$\mathcal{L}(N,v) = \frac{\mathbb{C}^p \times \mathbb{C}}{\Lambda^p} \quad \text{with } \lambda(x,u) = (x+\lambda, e_\lambda(x)u)$$

$$\downarrow \qquad \text{and}$$

$$E^p \qquad e_{n+m\tau}(x) = (-1)^{n^t N n}(-e^{i\pi\tau})^{m^t N m} e^{2\pi i m^t(Nx+v)}$$

The weight functions

$\Theta_k^+(z,y,\lambda)$ is the fiber of the vector bundle

$$\Theta^+_{k,n} \quad \text{on} \quad \hat{E}_T(pt).$$

Here $\Theta^+_{k,n} = (pr_2)_* \mathcal{L}(N^\Theta_{k,n}, 0)$ with $pr_2 : E^{\ell H} \times \hat{E}_T(pt) \to \hat{E}_T(pt)$

Then

$$\overline{\Theta}^\pm(z,y,\lambda) = \bigoplus_{k=0}^{n} \overline{\Theta}_k^\pm(z,y,\lambda) \quad \text{and}$$

$$\overline{\Theta}^\pm(z,y) = \bigoplus_{k=0}^{n} \overline{\Theta}_k^\pm(z,y)$$

The <u>weight</u> <u>functions</u> $\omega_I^\pm(t;z,y,\lambda)$ form a basis of $\overline{\Theta}_k^\pm(z,y,\lambda)$

$$\omega_I^\pm(t;z,y,\lambda) = \overline{\Phi}^\pm(z,y,\lambda)\, v_I, \quad \text{for } I \subseteq \{1,2,\ldots,n\}, \ \mathrm{Card}(I) = k.$$

The <u>normalized weight functions</u> are

$$W_I^-(t;z,y,\lambda) = \frac{\omega_I^-(t;z,y,\lambda)}{\prod_{j\neq \ell} \theta(t_j - t_\ell + y)}$$

$$W_I^+(t;z,y,\lambda) = \frac{\omega_I^+(t;z,y,\lambda)}{\Psi_I(y,\lambda) \prod_{j\neq \ell} \theta(t_j - t_\ell + y)}$$