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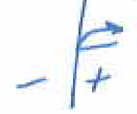
Level 0 representations and Macdonald polynomials

Math Physics Seminar University of Melbourne

Macdonald polynomials: Path formulas

positive fold $\gamma + k\delta$



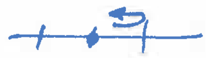
negative fold $\gamma + k\delta$

$$wt(f) = \frac{t^{-\frac{1}{2}}(1-t)}{1-q^k t^{ht(\gamma)}} \qquad wt(f) = \frac{t^{\frac{1}{2}}(1-t)\,q^k t^{ht(\gamma)}}{1-q^k t^{ht(\gamma)}}$$

$$E_\mu = \sum_{p \in B(\vec{\mu})} X^{end(p)}\, t^{\pm\frac{1}{2}}_{\varphi(p)} \prod_{f \in fold(p)} wt(f)$$

$$P_\lambda = \sum_{p \in P(\vec{\lambda})} X^{end(p)}\, t^{\frac{1}{2}}_{\varphi(p)} \prod_{f \in fold(p)} wt(f)$$

Examples: sl_2

$E_{2\omega} =$  $+$  $= t^{\frac{1}{2}} X^{\omega} + t^{\frac{1}{2}}\dfrac{(1-t)q}{1-qt}$

$E_{-2\omega} =$  $+$  $+$  $+$ 

$$= X^{-2\omega} + \frac{1-t}{1-qt} + \frac{1-t}{1-q^2t}X^{2\omega} + \frac{1-t}{1-q^2t}\,\frac{(1-t)q}{1-qt}$$

$P_{2\omega} =$  $+$ 

$$= X^{-2\omega} + X^{2\omega} + \frac{1-t}{1-qt} + \frac{(1-t)q}{1-qt}$$

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Lie algebras

$\mathfrak{g}_{af} = \mathfrak{g}[z, z^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ affine Lie algebra

$\cup|$

$\mathfrak{k} = \mathbb{C}K \oplus \mathbb{C}d \oplus \mathfrak{g}[z]$ current algebra

$\cup|$

$\mathfrak{b}_{af} = \mathbb{C}K \oplus \mathbb{C}d \oplus \mathfrak{b} \oplus z\mathfrak{g}[z]$ Iwahori-Borel subalgebra

$\cup|$

$\mathfrak{h}_{af} = \mathfrak{a}(z)$ half Heisenberg subalgebra

Enveloping algebras

$U\mathfrak{g}_{af}$ gen by $e_0, e_1, \ldots, e_n, f_0, f_1, \ldots, f_n, h_0, h_1, \ldots, h_n, d$

$U\mathfrak{k}$ gen by $e_0, e_1, \ldots, e_n,\ f_1, \ldots f_n, h_0, h_1, \ldots, h_n, d$

$U\mathfrak{b}_{af}$ gen by $e_0, e_1, \ldots, e_n$ $h_0, h_1, \ldots, h_n, d$

$\mathbb{C}[A] = U\mathfrak{h}_{af} = \mathbb{C}[h_{11}, h_{12}, \ldots] \otimes \mathbb{C}[h_{21}, h_{22}, \ldots] \otimes \cdots \otimes \mathbb{C}[h_{n1}, h_{n2}, \ldots]$

Affine braid/Weyl group gen by $T_0, T_1, \ldots, T_n$ acts by automorphisms $T_i : U_q\mathfrak{g}_{af} \to U_q\mathfrak{g}_{af}$

Integrable modules M are stable under the affine braid group action, ie.

$$T_i^* M \subseteq M.$$

M is level 0 if $K \cdot m = 0$, for $m \in M$.

Level 0 integrable modules

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Modules		Crystal	Character
Extremal weight	$T(\lambda)$	$\mathbb{B}^{\frac{\infty}{2}}(\lambda)$	$\left(\prod_{i=1}^{n}\prod_{k=1}^{\langle\lambda,\alpha_i^\vee\rangle}\frac{1}{1-q^k}\right)E_{-w_0\lambda}(q^{-1};\infty)$.
Extremal Demazure	$T(\lambda)_{\geq w}$	$\mathbb{B}^{\infty/2}(\lambda)_{\geq w}$	$\left(\prod_{i=1}^{n}\prod_{k=1}^{\langle\lambda_w,\alpha_i^\vee\rangle}\frac{1}{1-q^k}\right)E_{-w\lambda}(q^{-1};\infty)$ where $\lambda_w=\lambda-\sum_{w\alpha_j<0}\omega_j$.
local extremal weight	$T(\lambda;p)$	$\mathbb{B}_0^{\infty/2}(\lambda)$	$E_{-w_0\lambda}(q^{-1};\infty)$
local Demazure	$T(\lambda;p)_{\geq w}$	$\mathbb{B}_0^{\infty/2}(\lambda)_{\geq w}$	$E_{-w\lambda}(q^{-1};\infty)$

Terminology

Global Weyl module $T(\lambda)_{\geq w_0}$

Local Weyl module $T(\lambda;p)_{\geq w_0}$

Feigin Makyedonsyi module $T(\lambda;p)_{\geq w}$.

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Construction of the modules

$$\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n + 0\Lambda_0, \text{ with } \lambda_i \in \mathbb{Z}.$$

The extremal weight module $T(\lambda)$ is an integrable $U_q\mathfrak{g}_{af}$-module,

$$T(\lambda) = (U_q\mathfrak{g}_{af})u_\lambda, \qquad T_w^*: T(\lambda) \xrightarrow{\sim} T(w\lambda)$$
$$u_\lambda \longmapsto u_{w\lambda}$$

and, for $w \in W_{af}$ and $i \in \{0, 1, \ldots, n\}$

$e_i u_{w\lambda} = 0$ and $f_i^{(\langle w\lambda, \alpha_i^\vee\rangle)} u_{w\lambda} = 0$, if $\langle w\lambda, \alpha_i^\vee\rangle \geq 0$,

$f_i u_{w\lambda} = 0$ and $e_i^{(-\langle w\lambda, \alpha_i^\vee\rangle)} u_{w\lambda} = 0$, if $\langle w\lambda, \alpha_i^\vee\rangle \leq 0$.

Global to local

$$\mathbb{C}[\mathbb{A}^{(\lambda)}] = \mathbb{C}[h_{11}, \ldots, h_{1\lambda_1}]^{S_{\lambda_1}} \otimes \cdots \otimes \mathbb{C}[h_{n1}, \ldots, h_{n\lambda_n}]^{S_{\lambda_n}}$$

acts freely on $T(\lambda)$

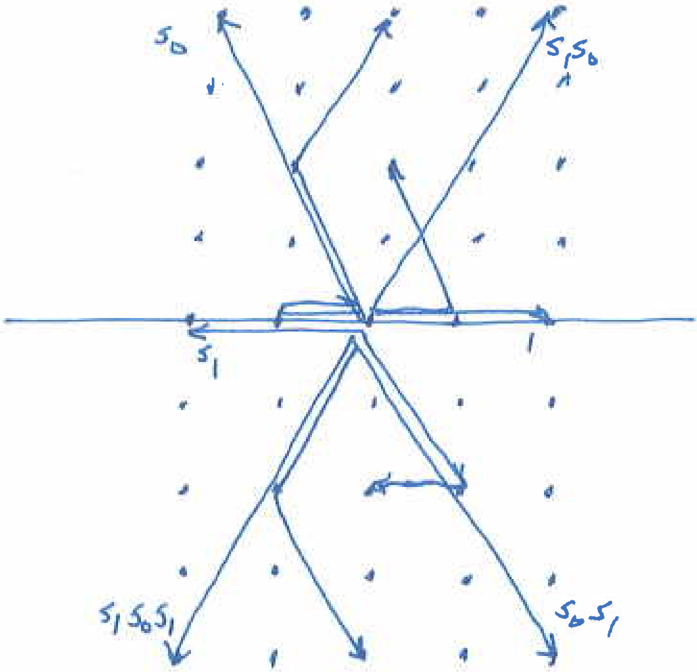
and
$$T(\lambda) \simeq \mathbb{C}[\mathbb{A}^{(\lambda)}] \otimes T(\lambda; 0)$$

and
$$B^{\frac{\infty}{2}}(\lambda) \simeq Par(\lambda) \otimes B_0^{\frac{\infty}{2}}(\lambda)$$

where $Par(\lambda) = \left\{ \vec{\rho} = (\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(n)}) \;\middle|\; \begin{array}{l} \rho^{(j)} \text{ partitions with} \\ \ell(\rho^{(j)}) \leq \lambda_j \end{array} \right\}$

The crystal $\mathbb{B}^{\frac{\omega}{2}}(2\omega)$ for $\mathfrak{sl}_2$

$p = \phi$



$\mathbb{B}_0^{\frac{\omega}{2}}(2\omega)$

$$E_{2\omega}(q,\infty) = X^{\omega} + 1$$

$$E_{-2\omega}(q,\infty) = X^{-2\omega} + q^{-1} + q^{-2}X^{2\omega} + q^{-2}$$

$$P_{2\omega}(q,\infty) = X^{-2\omega} + X^{2\omega} + q^{-1} + 1.$$

$p = \square$