Mendelssohn Salon 1828
Elliptic functions, Kosmos and Beethoven

A public lecture of Michael Leslie and Arun Ram

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Berlin 1828: The salons of Abraham and Lea Mendelssohn were the foci for some of the greatest scientific and musical minds of the day. Alexander von Humboldt, famous for his scientific findings from his voyage to the wild Americas, had set up a hut for measuring the earth's magnetic field in the Mendelssohn garden. In the previous year Beethoven had died leaving a controversial musical legacy, while Abel and Jacobi created a mathematical revolution between 1827 and 1829 with the development of elliptic functions.

The discussion and debate this evening will be music and mathematics: the sounds of Beethoven, the elliptic orbits of the planets, and the thrill of the Kosmos.

Arun Ram and Michael Leslie hope to recreate the passion and exuberance which were the hallmark of those legendary gatherings.

Snout: *Doth the moon shine that night we play our play?*
Bottom: *A aalendar, a calendar! Look in the almanac, find out moonshine, find out moonshine.*
Quince: *Yes, it doth shine that night.*

-- Shakespeare, from "Midsummer Night's Dream".

Der Mensch ist nur da ganz wo er spielt -- Schiller

This above all: to thine ownself be true,
And it must follow, as the night the day,
Thou canst not then be false to any man.

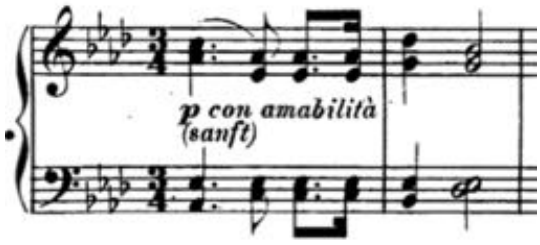
-- Shakespeare, Polonius to Laertes in "Hamlet"

momentary as a sound
swift as a shadow
short as any dream
brief as the lightning in a collied night that, in a spleen, unfolds both heaven and earth
and ere a man has power to say "Behold!"
the jaws of darkness do devour it up.

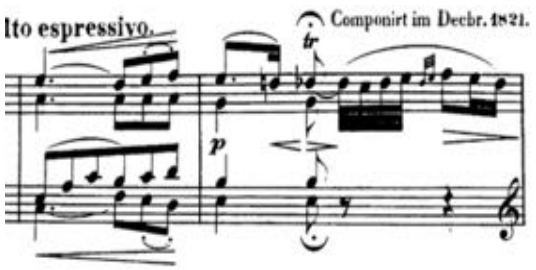
-- Shakespeare, from "Midsummer Night's Dream".

Alexander von Humboldt

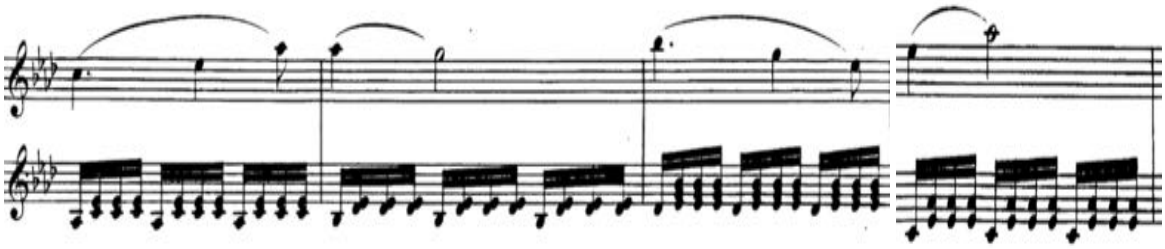
Berlin April 1828: It was an unusual time. (I 1-2 **Opening 2 bars**)



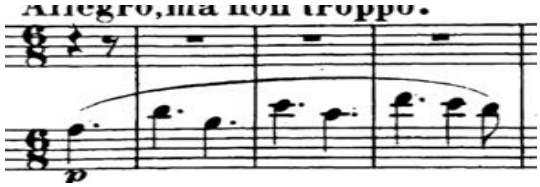
The sun was shining. (I 3-4 **continuation: flowering and trill and speak through**)



But was so striking was the way that the sun was shining in people's imaginations in such a new inspiring way. (I 5-9)

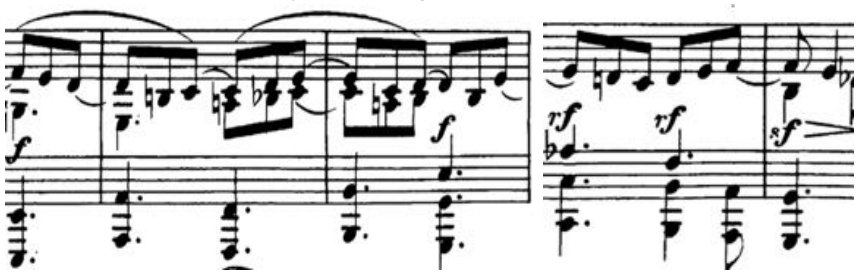


Alexander von Humboldt had just finished a series of 62 lectures at the University of Berlin which were bringing the students to a new vantage point of the universe. (III 27 **Fugue theme**)

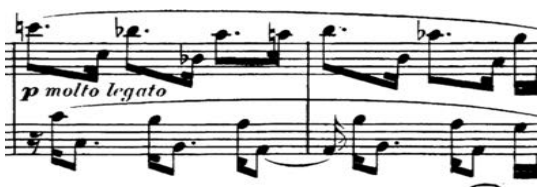


Fanny Mendelssohn wrote to the diplomat Klingemann, who was a close friend of the family

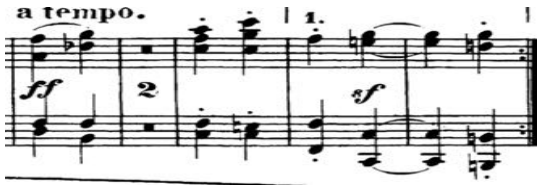
> "You may perhaps know that Alexander von Humboldt is giving a
> course at the university (in physical geography),
> but did you also know that after an entreaty from the highest quarters
> he has begun a second course in the hall of the Singakademie,
> attended by everyone who claims to be educated and fashionable
> — everyone, from the king and all his court down through the
> ministers, generals, officers, artists, scholars, writers,
> wits and dullards, arrivistes, students, and women,
> even including your present unworthy correspondent?
> A frighteningly large crowd throngs to them, the audience is imposing,
> and the course is infinitely interesting." (III 101-105)



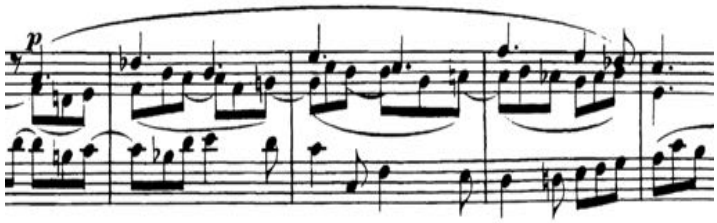
As Fanny explains, the lectures were so well received
that Humboldt gave an additional 16 lectures "for the public".
Humboldt was the voice of science to the public.
He was the first Carl Sagan.
The first David Attenborough.
He inspired the imagination with fact and science and discovery as had never been done before him.
He told of stars, the universe, nebulae, (I 20 **white birds circling**)



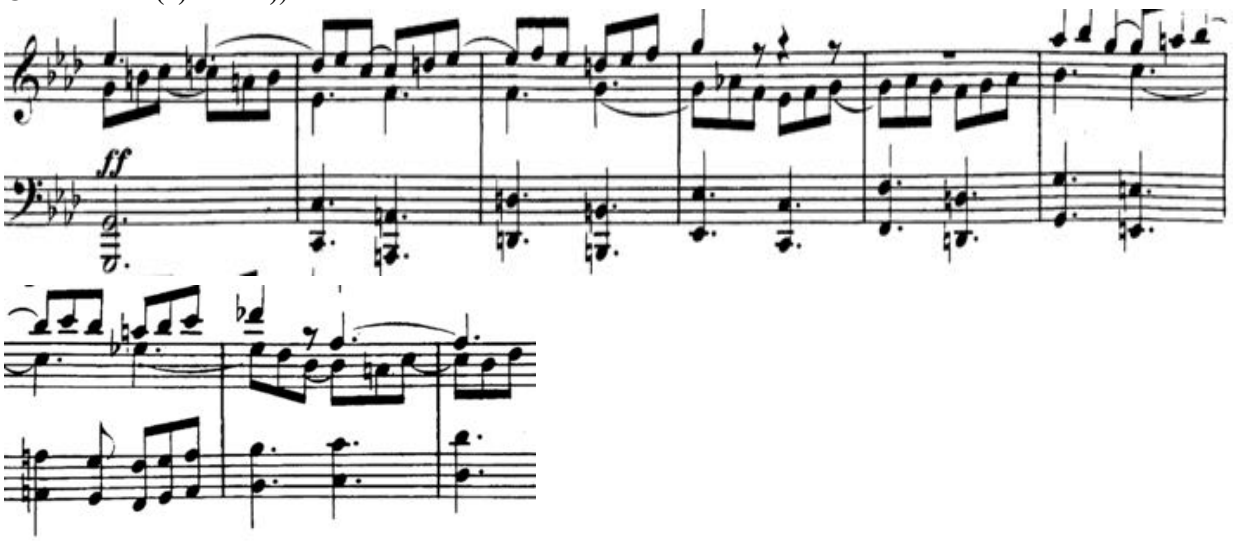
the brilliance of meteor showers, the motion of the planets, the explosions of volcanoes, (II 36 **scherzo fortes**)



the earth's magnetic fields, the sources of rocks and minerals, (III 36 **Fugue 1 3rd entry A ♭**)



and the births and deaths of mountain ranges, islands, and continents by motion of tectonic plates. (III 73 **Fugue 1 Octave bass (1) 4 bars)**)

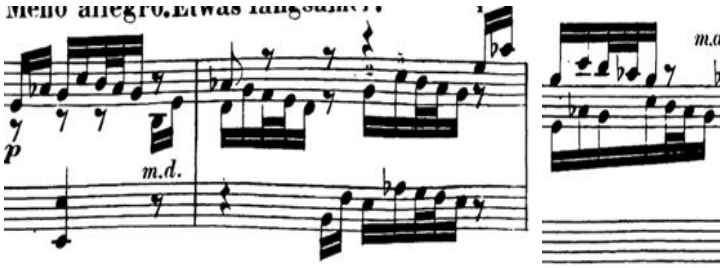


Humboldt became famous from his multivolume book with Bonpland,
“Personal Narrative of a Journey to the Equinoctal Regions of the New Continent”.
It was the published output of a 5 year voyage to South America, Mexico and North America.
It is thrilling and exciting. Let me read a bit to you.

> the gymnoti are electric eels which "inhabit the Rio Colorado, the Guarapiche,
> and several little streams which traverse the Missions of the Chayma Indians.
> In the Llanos, between the farms of Morichal and the Upper and Lower Missions,
> the basins of stagnant water and the confluents of the Orinoco are filled with electric eels.
> We at first wished to make our experiments in the house we inhabited at Calabozo;
> but the dread of the shocks caused by the gymnoti is so great that,
> though they are easily caught,
> and we had promised the Indians two piastres for every strong and vigorous fish,
> during three days we could not obtain one.
>
> Impatient of waiting, we repaired to the Cano de Bera,
> to make our experiments in the open air, and at the edge of the water.
> We set off on the 19th of March, at a very early hour, for the village of Rastro;
> thence we were conducted by the Indians to a stream, which,
> in the time of drought, forms a basin of muddy water, surrounded by fine trees, the clusia, the amyris,
> and the mimosa with fragrant flowers.
> To catch the gymnoti with nets is very difficult, on account of the extreme agility of the fish, which
> bury themselves in the mud.

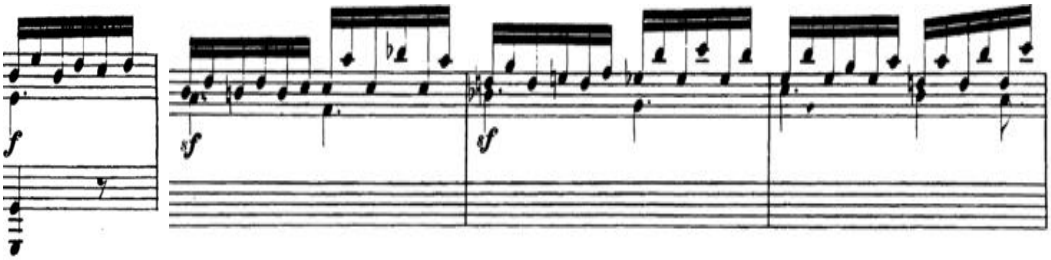
We would not employ the roots of the Piscidea erithyrna,
which thrown into the pool, intoxicate or benumb the eels.
These methods have the effect of enfeebling the gymnoti.
The Indians therefore told us that they would "fish with horses,"
We found it difficult to form an idea of this extraordinary manner of fishing;
but we soon saw our guides return from the savannah,
which they had been scouring for wild horses and mules.
They brought about thirty with them, which they forced to enter the pool.

The extraordinary noise caused by the horses' hoofs,
makes the fish issue from the mud, and excites them to the attack.
(III 168 **eel music**)

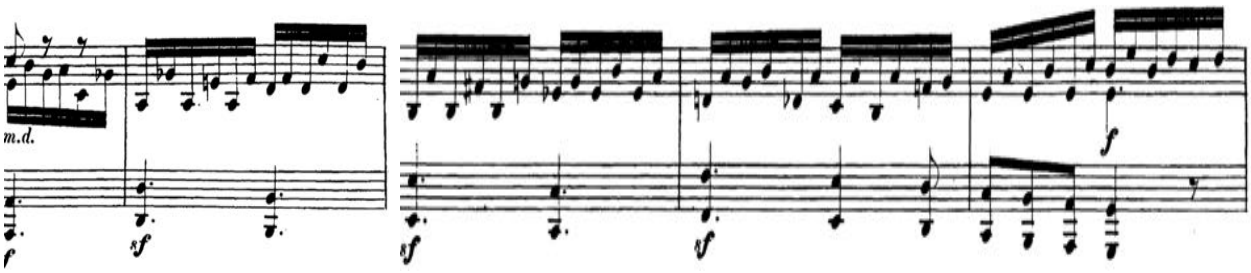


These yellowish and livid eels, resembling large aquatic serpents,
swim on the surface of the water, and crowd under the bellies of the horses and mules.
(III 170.5 **continued eel music**)

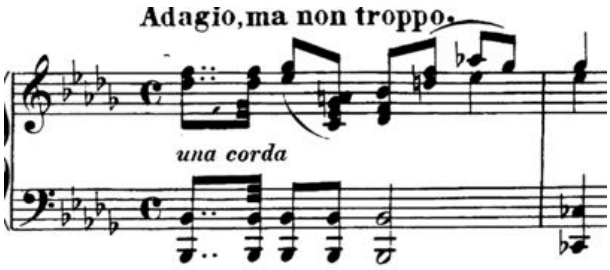
The Indians, surround the pool closely; and some climb up the trees,
the branches of which extend horizontally over the surface of the water.
By their wild cries, and long reeds,
they prevent the horses from running away and reaching the bank of the pool.
The eels, stunned by the noise, defend themselves by the repeated discharge of their electric batteries.
For a long interval they seem likely to prove victorious. (III 178.5)



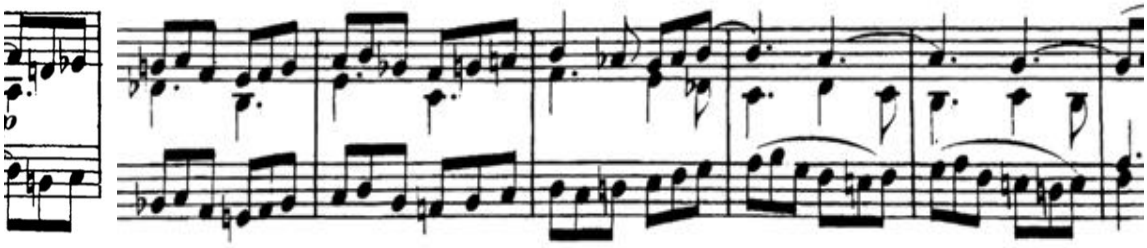
Several horses sink beneath the violence of the invisible strokes which they receive from all sides,
and stunned by the force and frequency of the shocks, they disappear under the water.
Others, panting, with mane erect, and eyes expressing anguish, raise themselves, and endeavour to flee from the storm by which they are overtaken. (III 174 **beginning of rhapsody**)



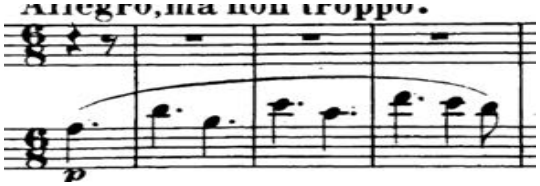
They are driven back by the Indians into the middle of the water;
but a small number succeed in eluding the active vigilance of the fishermen.
These regain the shore, stumbling at every step,
and stretch themselves on the sand, exhausted with fatigue,
and with limbs benumbed by the electric shocks of the gymnoti. (III 1 **opening of 3rd mvmt**)



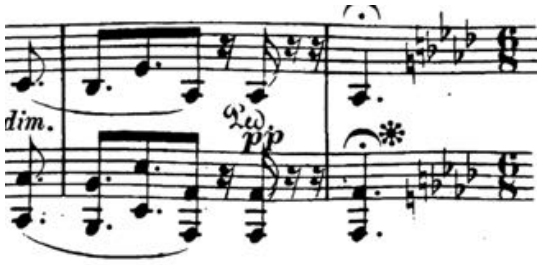
We had little doubt that the fishing would terminate by killing successively all the animals engaged;
but by degrees the impetuosity of this unequal combat diminished, and the wearied gymnoti dispersed.
They require a long rest, and abundant nourishment, to repair the galvanic force which they have lost.
The mules and horses [begin to] appear less frightened; their manes are no longer bristled, and their eyes express less dread.
The gymnoti approach timidly the edge of the marsh, where they are taken by means of small harpoons fastened to long cords.
When the cords are very dry the Indians feel no shock in raising the fish into the air.
In a few minutes we had five large eels, most of which were but slightly wounded. (III 63 **piano, last entry of fugue theme BEFORE crashing bass octaves**)



We found that a fish of three feet ten inches long weighed twelve pounds.
The transverse diameter of the body was three inches and a half.
The gymnoti of the Cano de Bera are of a fine olive-green.
The under part of the head is yellow mingled with red.
Two rows of small yellow spots are placed symmetrically along the back, from the head to the end of the tail. (III 27 **Fugue theme**)

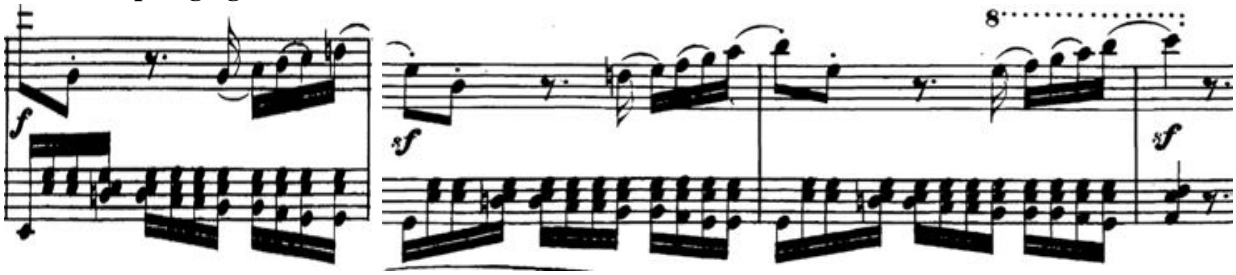


Every spot contains an excretory aperture.
In consequence, the skin of the animal is constantly covered with a mucous matter, which,
as Volta has proved, conducts electricity twenty or thirty times better than pure water.
It is in general somewhat remarkable, that no electric fish yet discovered in the different parts of the world, is covered with scales. (III 25 **last two bars of klagender gesang -- final cadence**)

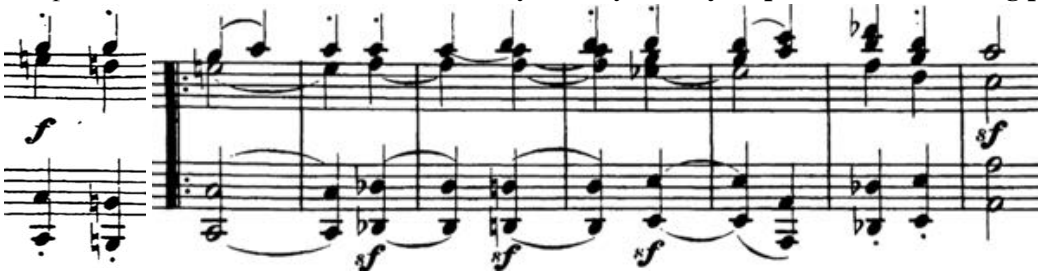


The Mendelssohns

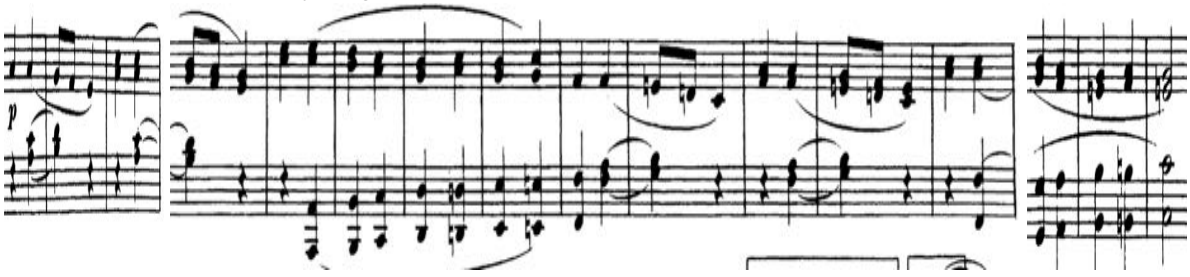
Humboldt's books, his infinite curiosity and his inspiring interest in everything and everyone
had made him the centre of scientific knowledge at the time, more famous than Einstein is now. (I 28 **plunging treble and plunging bass chords**)



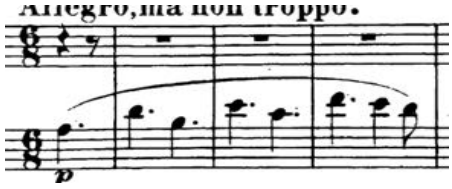
What was the right place to find Humboldt around town?
Well, of course, at the Mendelssohn household.
Joseph Mendelssohn was a banker, immensely wealthy. (II 9 **syncopations in connecting passage scherzo**)



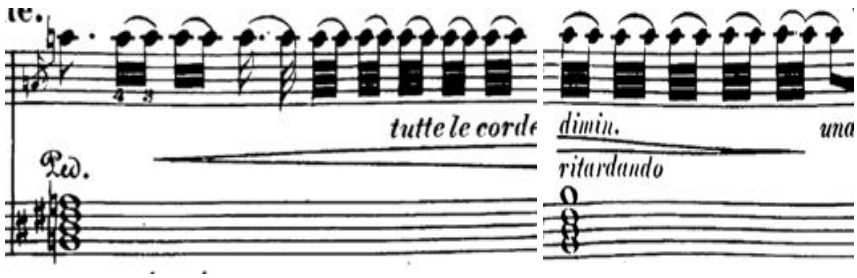
He had helped to fund von Humboldt's scientific expedition to the Americas.
But it was the household of Joseph's brother, Abraham Mendelssohn and his wife Lea,
that was the centre of everything intellectual at the time. (II 17 **ich bin liederlich**)



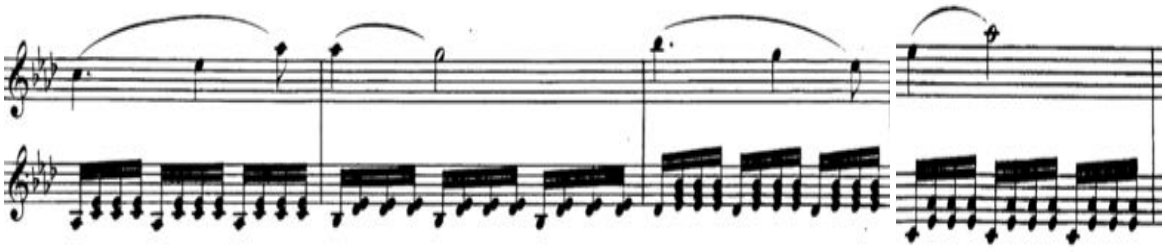
And since Humboldt was the centroid of science, he was, naturally, a familiar feature in the Mendelsssohn household. (III 27 **Fugue theme**)



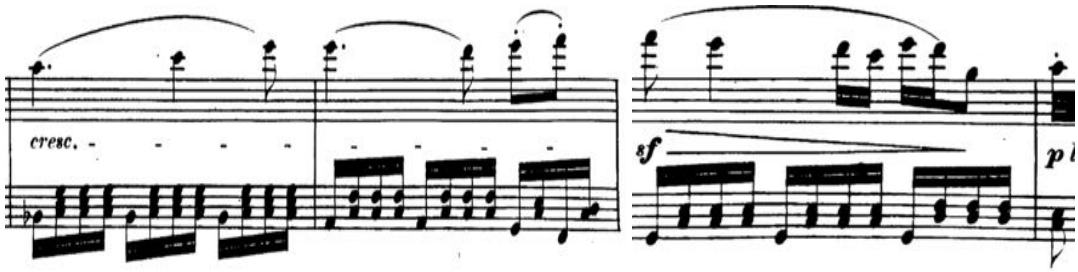
Humboldt and the great mathematician astronomer Encke had set up a special laboratory
"the hut" in the Mendelssohn's garden in order to make the most precise measurements of the Earth's magnetic field that had ever been made. (III 5 **repeated a flats**)



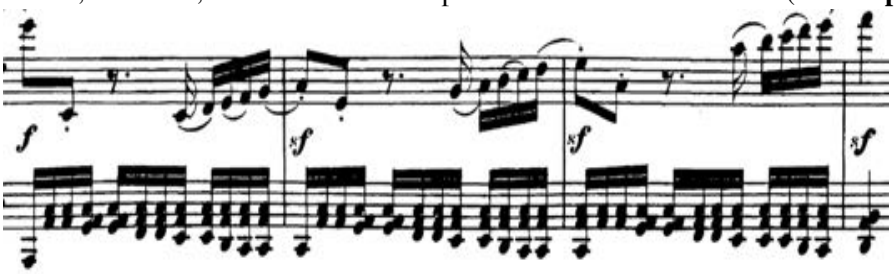
And over in the garden wing of the Mendelssohn house, the Mendelssohn children were industrious. (I 5 **and speak through**)



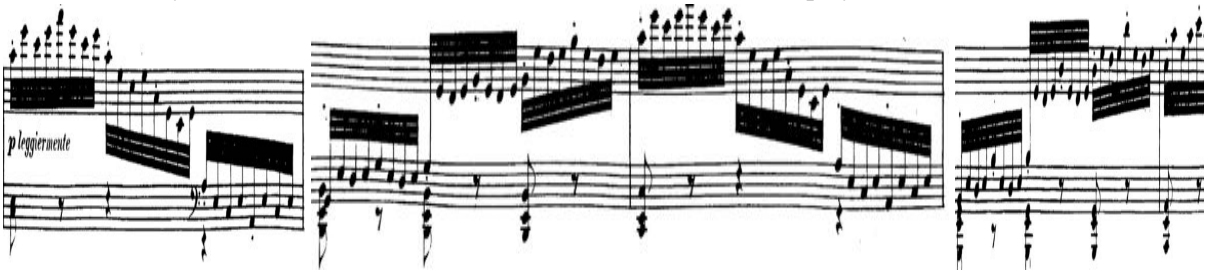
A few summers before they had been reading Shakespeare's Midsummer's Night Dream. (I 8 **continuation**)



At that time Shakespeare was hardly the bread and butter of children's schooling.
If Goethe hadn't proclaimed Shakespeare's greatness at the first German "Shakespeare Day" not long before,
then probably nobody in the German community would have had more than the most minimal recognition of the name.
It was the Schlegel-Tieck translations,
done by the brother-in-law of the children's aunt Dorothea that spread these plays all over Germany
and so, of course, the children were in possession of the first edition. (I 87-90 **plunging bit**)



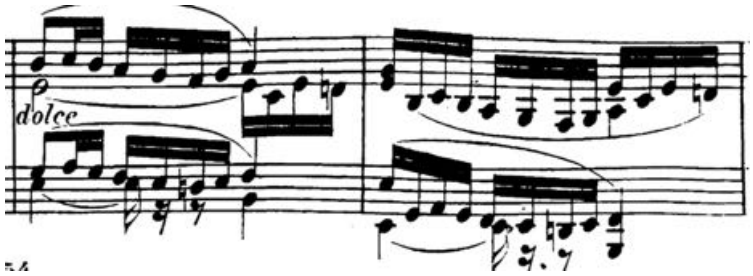
The kids, Fanny, Felix, Rebecca and Paul, had a brilliant time with the play. (I 12 **Rebecca's theme**)



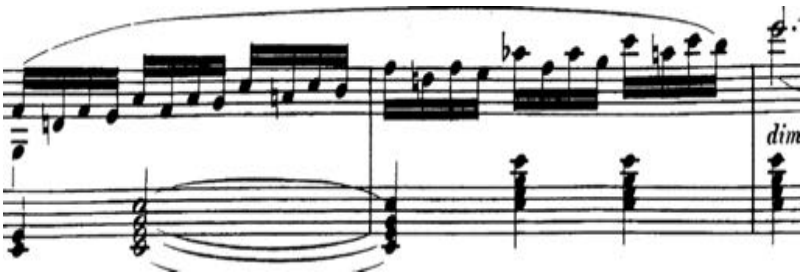
Fanny was the eldest. She naturally had to play Hippolyta.

> "Four days will quickly steep themselves in night.
> Four nights will quickly dream away the time.
> And then, anon, like to a silver bow bent in heaven, shall behold the night of our solemnities."

Wilhelm Hensel, the royal court artist,
was often around and stepped into the part of Theseus (I 34-35 **Hensel's theme resolved**)



Hippolyta, I woo'd thee with my sword, and won thy love doing thee injuries.
But I wed thee in another key. With pomp, with triumph and with reveling." (I 36 **continuation**)



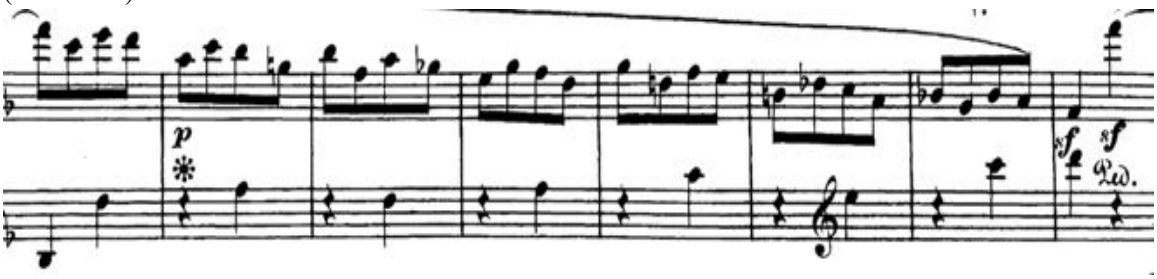
Hensel was truly a brilliant artist.
His legacy of vivid portraits bring to life many important persons of the time.

The youngest of the Mendelssohn children, Paul, took the part in the play of the parent, not unlike his own mother,

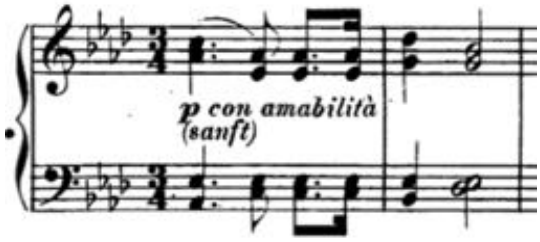
who could not easliy accept the obvious chemistry between Fanny and Hensel.

> "This man has bewitched the bosom of my child,
> thou hast given her rhymes and interchanged love tokens with my child,
> Thou hast by moonlight at her window sung with feigning voice verse of feigning love,
> and stolen the impression of her fancy with bracelets of thy hair,
> with rings, gauds, conceits, knacks, trifles, nosegays, sweetmeats,
> messengers of strong prevailment in unhardened youth [let me tell you],
> with cunning thou hast filched my daughters heart."

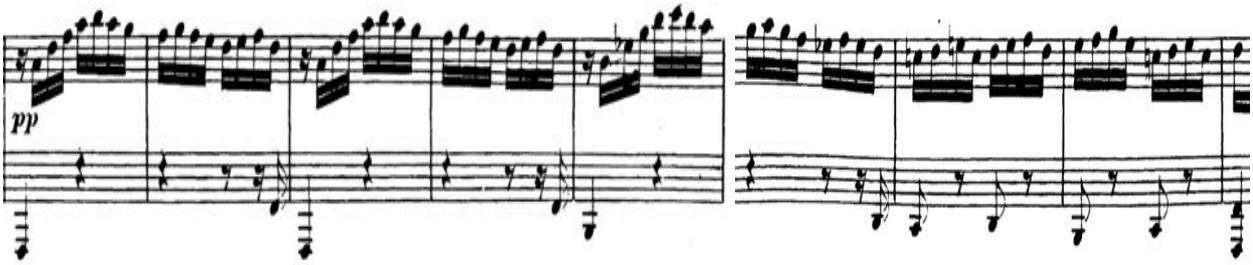
(II 41 **Trio**)



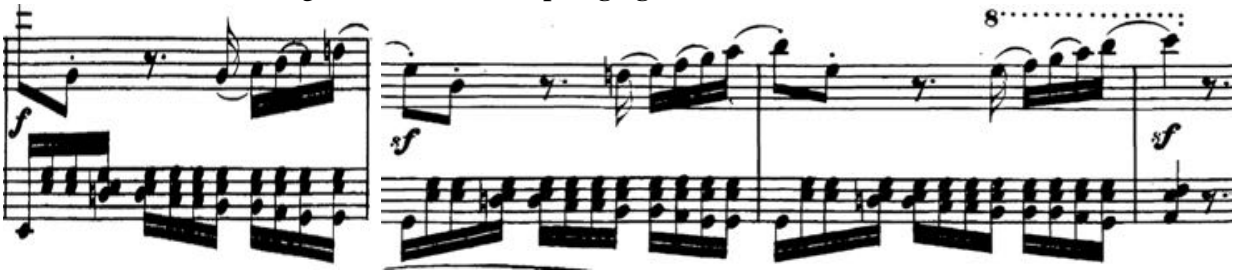
But the children's real focus was music. (I 1 **opening of sonata**)



The Mendelssohn parents cut no corners in making sure that they had had the very best teachers.
Madame Bigot was their first main piano teacher.
Beethoven was apparently so impressed with her reading of the "Appasionata sonata" that he later presented her with the manuscript. (Appasionata **up and down twice**)

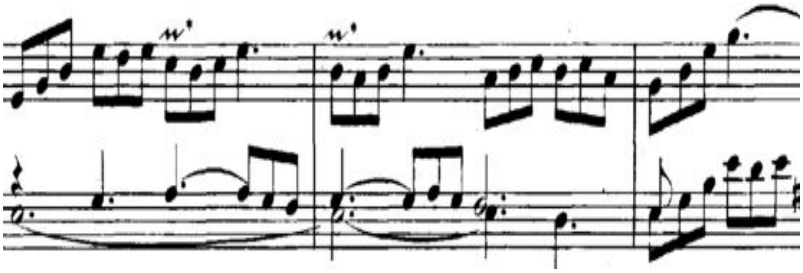


Madame Bigot did much to introduce Beethoven's piano music to Parisian audiences.
She was the Nadia Boulanger of the time. (I 28 **plunging music**)



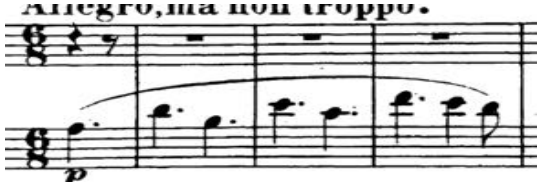
In Berlin, the children's primary teacher was Zelter, a friend and colleague of Goethe, who was director of the music conservatory.

When Fanny was 13 she played all the preludes from Bach's Well Tempered Klavier for her father on his birthday, from memory, of course. (Bach **E major Book I**)

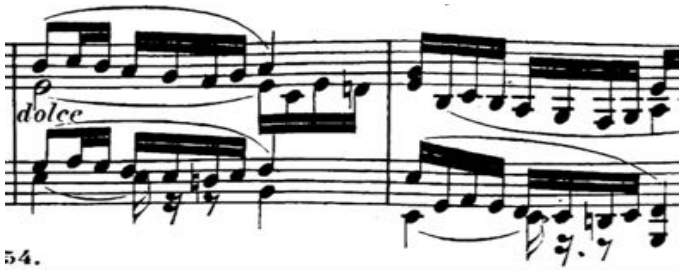


The guests

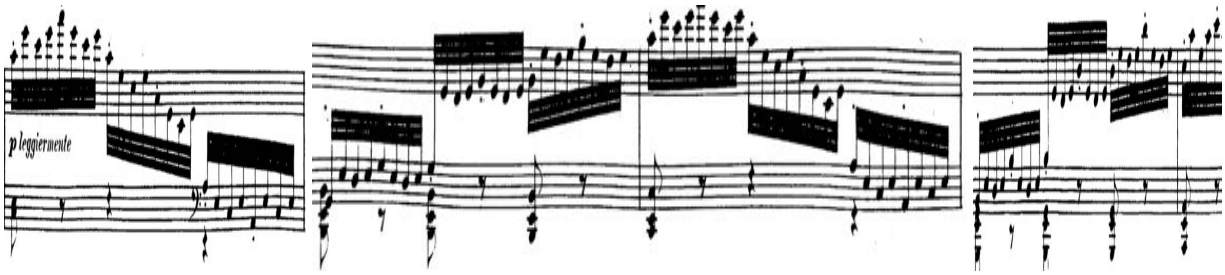
And with such musically talented children, it was not a problem to have the best music at the Salons that the Mendelssohn parents ran.
These salons not only hosted von Humboldt (III 27 **Fugue**),



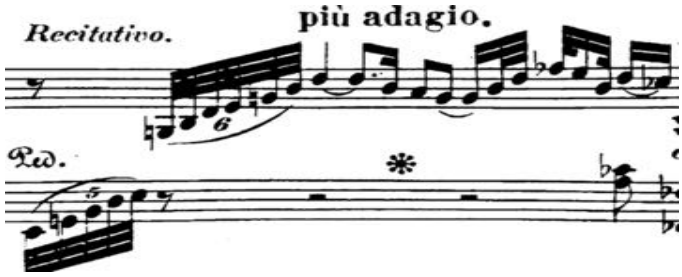
the royal court artist Wilhelm Hensel (I 34 **Hensel's cadence theme**),



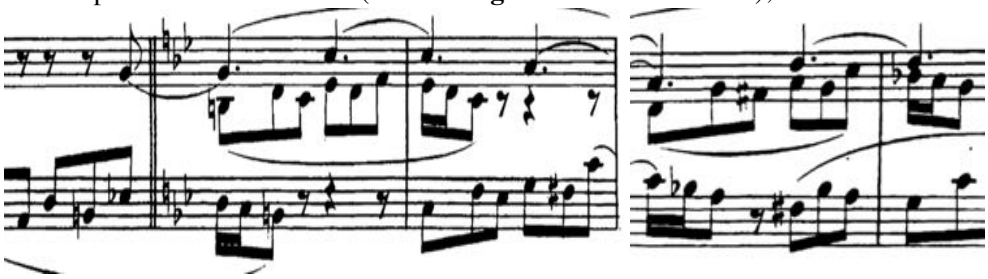
the violinist Paganini, (**24th caprice**)
the engineer Crelle, the diplomat Klingemann,
and several extremely talented young men who were all in love with Rebecka (I 12 **Rebecka's theme**)



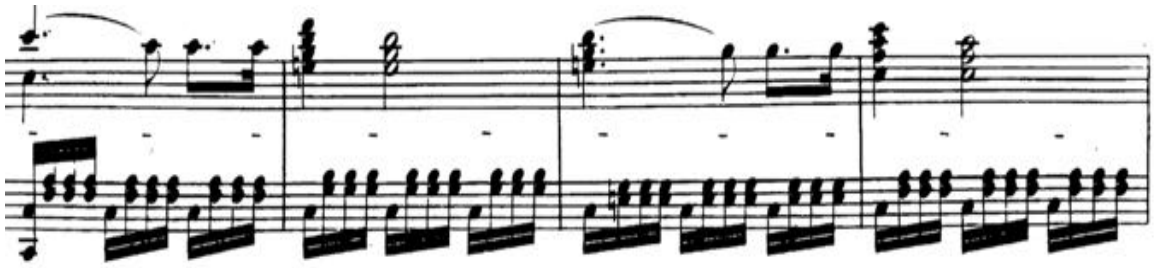
The young men included the melodramatic poet Heinrich Heine (III 4 **recitative**),



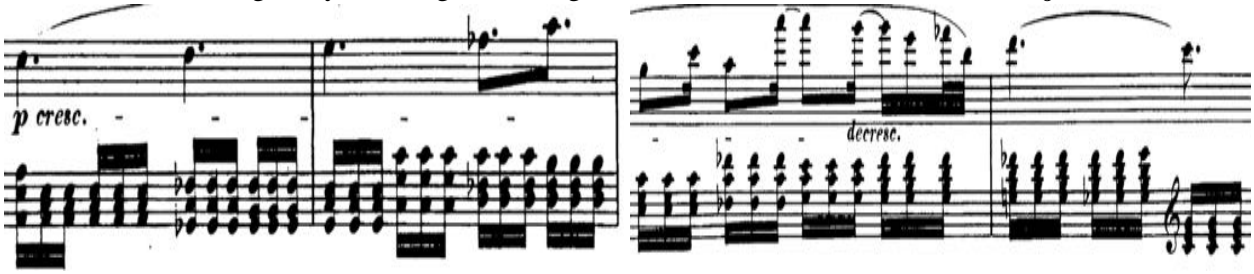
the law professor Eduard Gans (III 152 **Aug+Dim c minor 4 bars**),



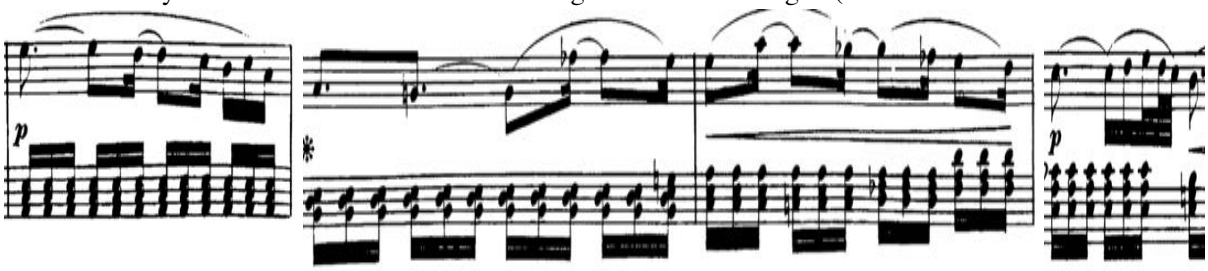
and the Greek/Roman classicist and historian Johann Droysen. (I 40 **4 bars f minor over Cnatural**),



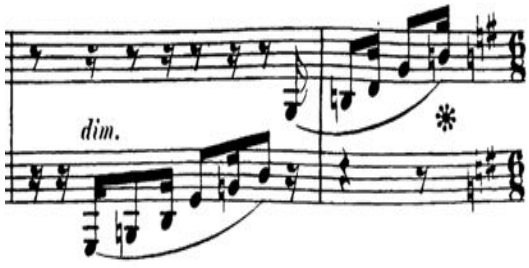
Rebecka was charming, witty, and sang like an angel. (III 13 **K.G. Aflat minor to Cflat major**)



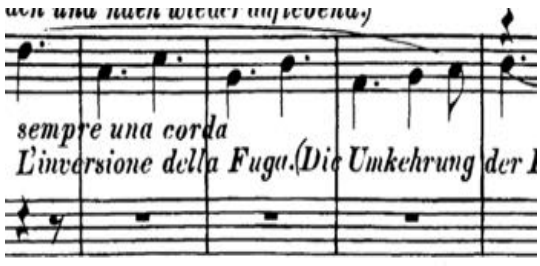
Fanny and Felix wrote beautiful songs,
and it was easy to fall in love with such a charming and attractive singer. (III 9-12 middle **K.G. A flat minor**)



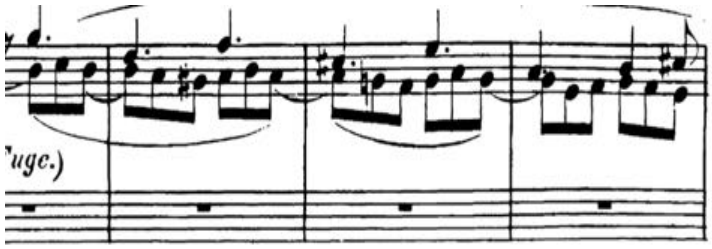
The mathematical tradition of 19th century Germany was tightly interconnected with the Mendelssohn family (III 135 **lead up to first fugue entry**),



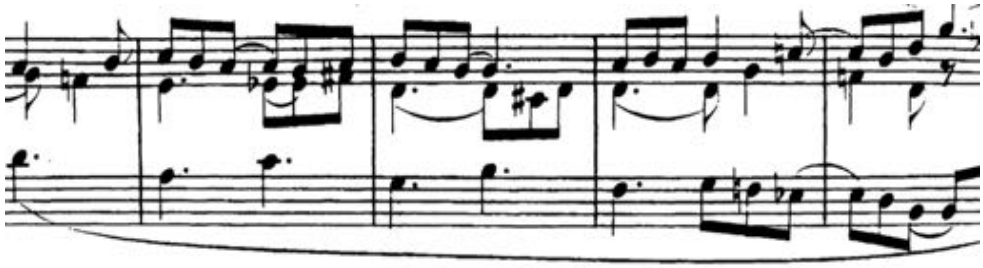
The talented artist Hensel was engaged to Fanny,
and they were to be the grandparents of Kurt Hensel.
The extremely useful and powerful Hensel's Lemma is one of the fundamental tools
for the most exciting new discoveries in number theory in the 2010s.
(III 137 **inversion of fugue, 1st entry**)



Little Paul's future brother-in-law, Eduard Heine (who was no relation to the poet Heinrich Heine),
is the Heine in the Heine-Borel theorem
which I will teach next week in my Metric and Hilbert spaces class
next door at Melbourne University.
(**inversion of fugue, second entry**)

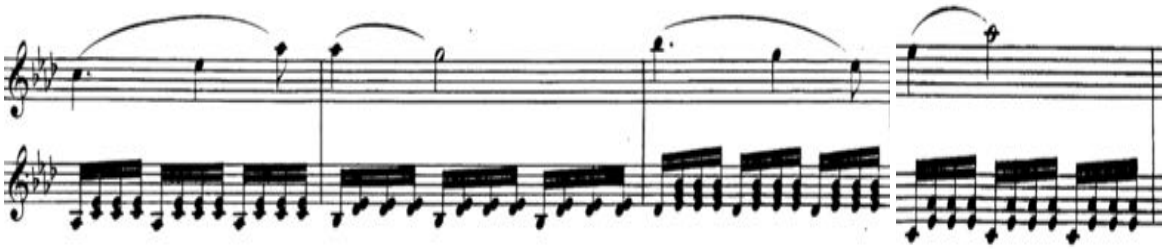


Ernst Eduard Kummer,
the number theorist who would lay fundamental foundations for the astounding achievements of Grothendieck
was the husband of Ottilie Mendelssohn,
the Mendelssohn children's first cousin. (**inversion of fugue third entry**)

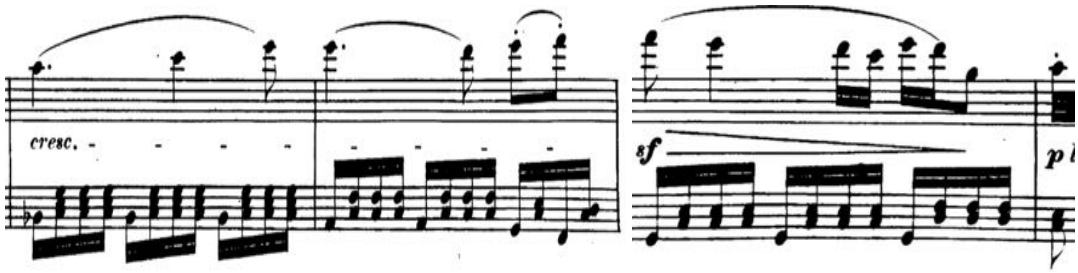


Dirichlet

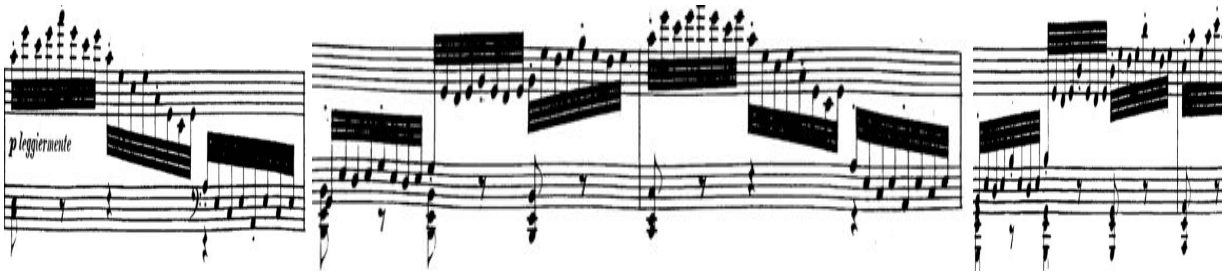
That particular Sunday, they were preparing for that afternoon's salon. (I 5 **and speak over to reach Rebecca**)



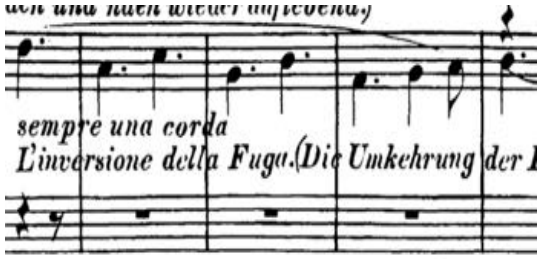
Felix and Fanny were readying the music for the afternoon.
Abraham was out discussing the placements of the pavilion with the gardner.



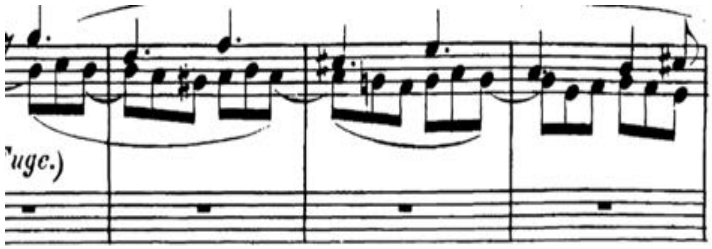
Lea was directing the flower arrangements and the food preparations. (and four bars)
And Rebecka was happily flitting in and out rehearsing her songs in her head. (I 12 **Rebecca theme**)



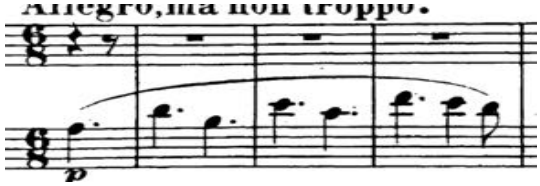
Among those invited that afternoon was Dirichlet,
Dirichlet was the son of a postmaster from a little town near Aachen which,
depending on the political situation, was either in France or Prussia.
Dirichlet was a star young mathematician, who had been educated in Paris. (III 137 **Inversion of fugue theme 1st entry**)



But Dirichlet felt German, especially in Paris, and wanted to move to Germany. (III 140 **inverted fugue second entry**)



But to move to Germany Dirichlet needed a job.
Fortunately the Prussian chief scientist, Humboldt (III 27 **first fugue theme**)

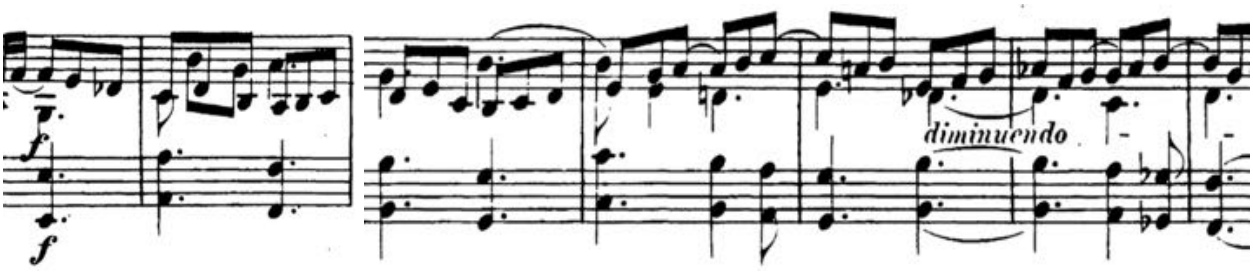


had met Dirichlet in Paris.
And Humboldt had checked up on him. Fourier and Poisson had given Dirichlet highest recommendation. (III 137 **inversion of fugue theme 1st entry**)

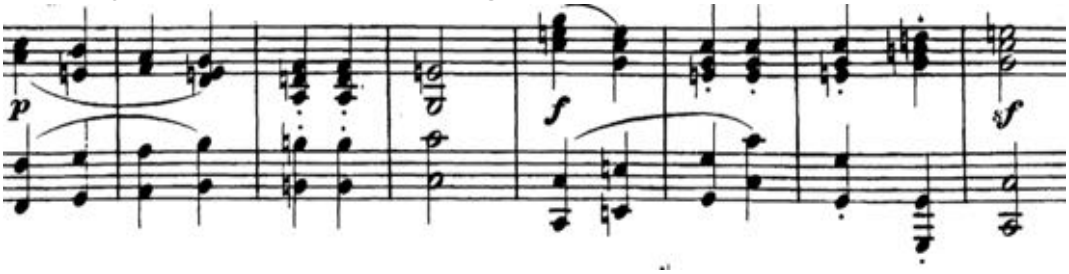
But to be sure that this guy was an absolutely a top mathematician

Humboldt wrote to Gauss, the greatest German mathematician.
Gauss, who never gave much praise, said he was an "outstanding talent"
and Bessel said

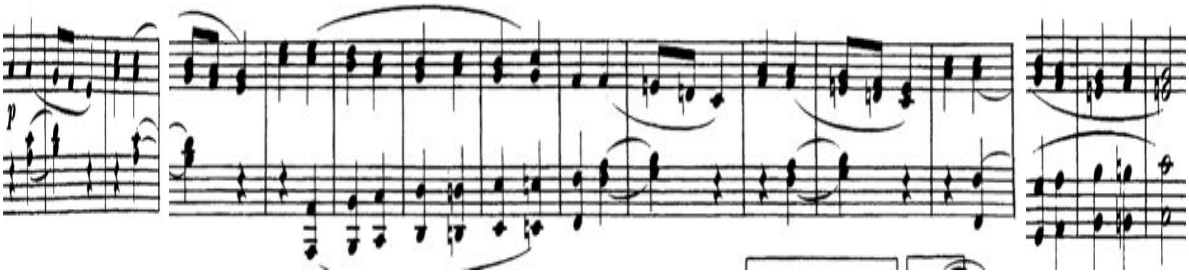
> "who could have imagined that this genius would succeed
> in reducing something seeming so difficult to such simple considerations.
> The name Lagrange could stand at the top of the memoir,
> and nobody would have guessed the error." (III 45.5 **first octaves fugue I**)



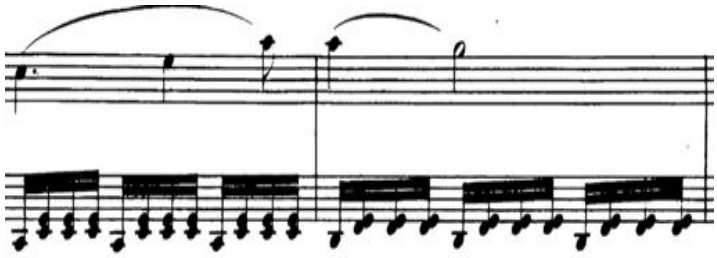
That Sunday afternoon, von Humboldt had invited Dirichlet to the Mendelssohn salon, and he was on his way, in the carriage, (II 17 **Unser Katz hat Katzl ghabt**)



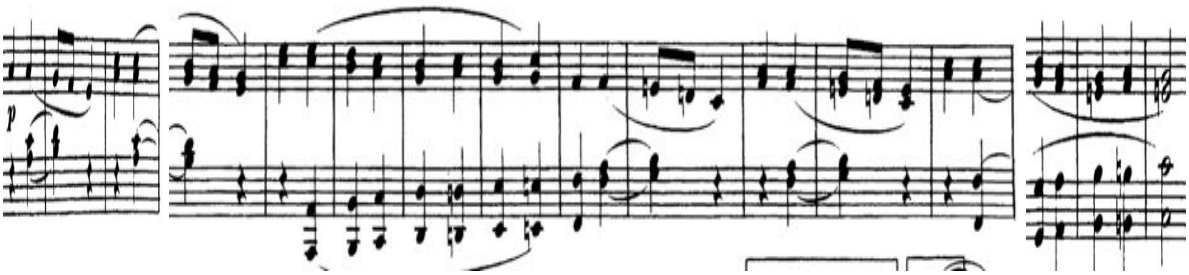
checking his tie, checking his watch, checking that his shoes weren't getting mud on them. (II 1-8 **Ich bin liederlich**)



Dirichlet became a familiar visitor to the Mendelssohn household. (I 5)



Fanny Mendelssohn wrote to a friend
The law professor "Gans is happy to scuffle with Dirichlet, the very handsome, very pleasant, and very learned mathematics professor, who looks like a student full of the joys of living". (II 17 **ich bin Liederlich**)



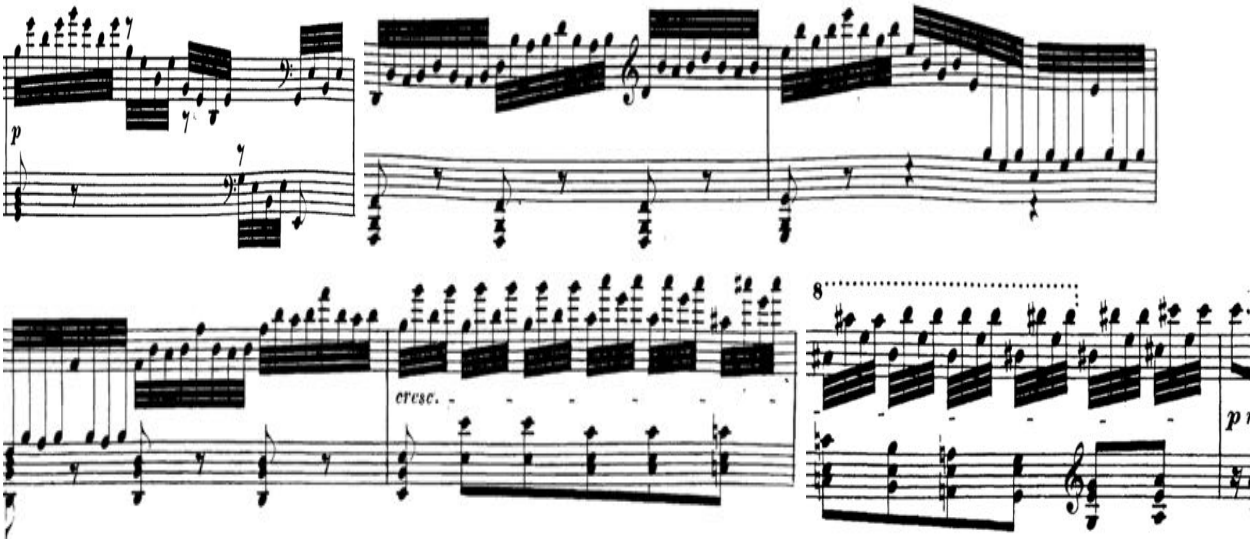
Indeed, in spite of those melodramatic poets, astute law professors, and refined classicists, all in love with Rebecca,

Heinrich Heine

Eduard Gans

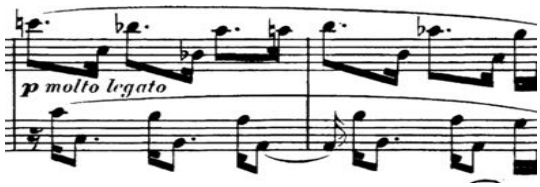
Johann Droysen

Dirichlet was the one that won Rebecka's attentions
and they were engaged 3 years later in 1831. (I 70 **Rebecca theme up and down twice**)

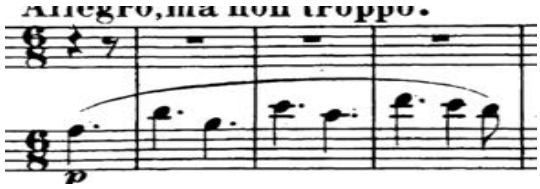


The concept of scale

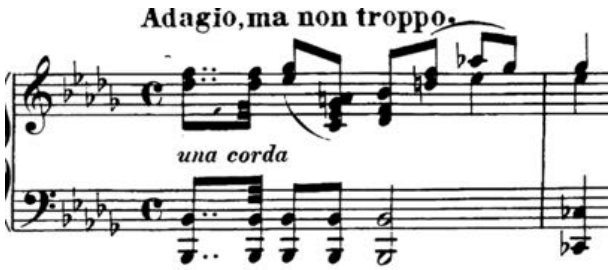
At the salon that afternoon the discussion was all about the motion of the planets,
the music of the celestial spheres and elliptic functions. (I 20 **white birds circling**)



Humboldt's book stimulated by the Berlin lectures of 1828 is entitled:
"Kosmos. A Sketch of a Physical Description of the Universe"
(III 27 **fugue theme solo**)

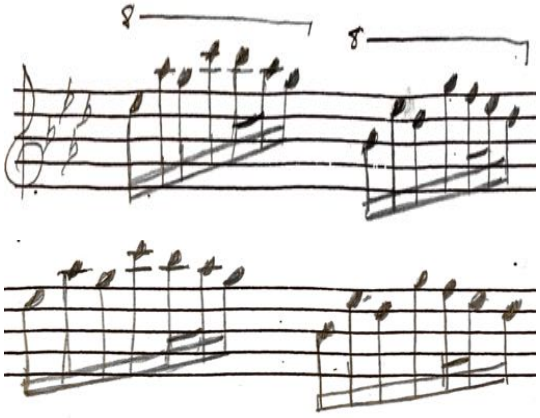


Even today it is inspiring, fascinating and extremely educational.
It lives the "scientific method" which we try to give a picture of in science classes.
Let me read a bit from Humboldt's Kosmos for you,
where he explores scale in our solar system (III 1 **beginning of 3rd movement**)



The planets nearest the Sun, in the inner group,
are of more moderate size, denser, rotate [around their axes] more slowly and with nearly equal velocity.

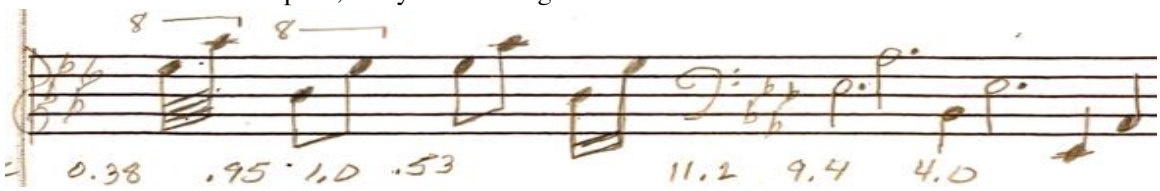
Andante



The exterior planets, which are further removed from the Sun,
are very considerably larger, have a density five times less, more than twice as great a velocity in the period of their rotation round their axes."

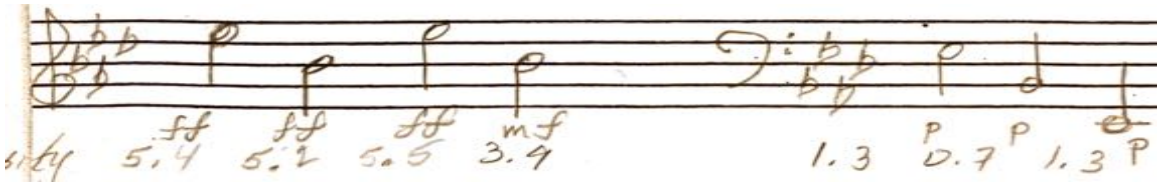
Presto

“Mars is smaller than the Earth and Venus,
approaching most nearly in size to Mercury, the nearest planet to the Sun.
Saturn is smaller than Jupiter, and yet much larger than Uranus.



“Uranus appears to be denser than Saturn,
and we find both Venus and Mars less dense than the Earth, which lies between them.”

8----------|



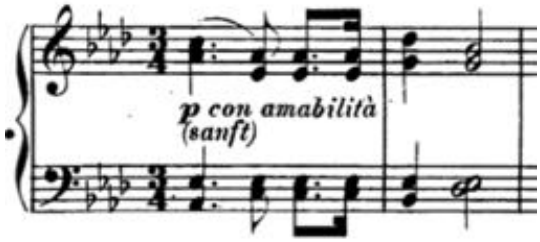
The elliptic orbit [of] Mercury [has] the greatest degree of eccentricity,
and Venus and the Earth, which immediately follow each other, have the least eccentricity."

Beethoven's great work, the piano sonata Op. 110
also has an exploration of scale, from small to large.
It comes in a sequence of keys, which you might think of as the musical planets:

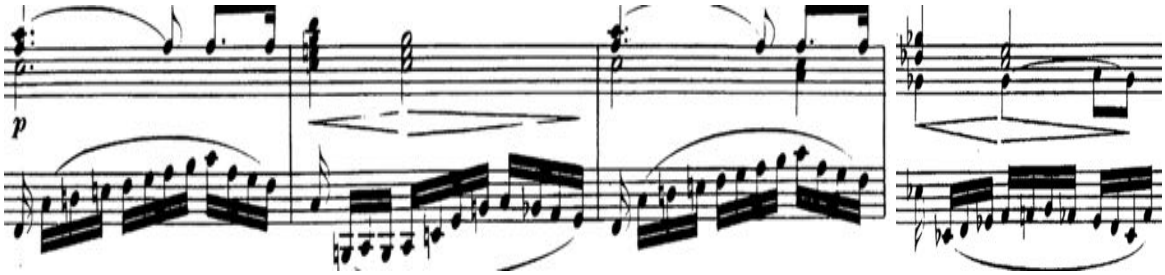
The home key is Aflat major,
Then down a third to F minor,
Then down a third to Dflat major,
Then down a third to Bflat minor,
and finally back to Aflat major.

This sequence of keys appears in the development section of the first movement, in rapid succesion,
all 5 keys, in seqeunce, in the space of 14 bars:

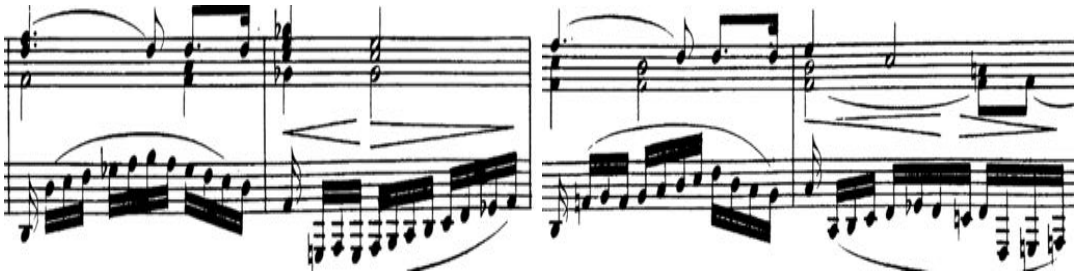
The home key is Aflat major,



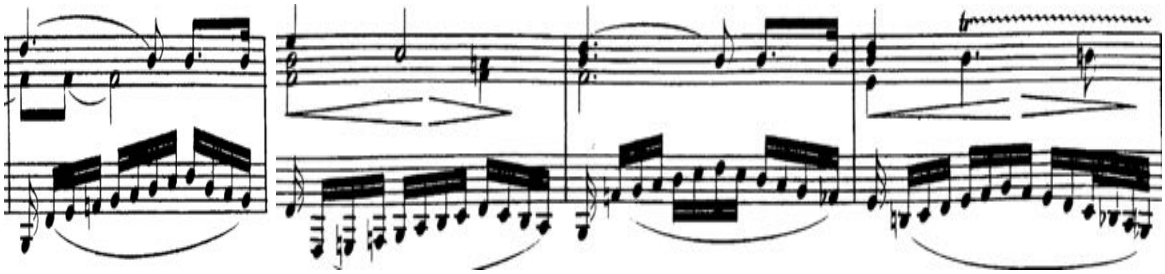
which, in the development section, moves down a third to F minor,



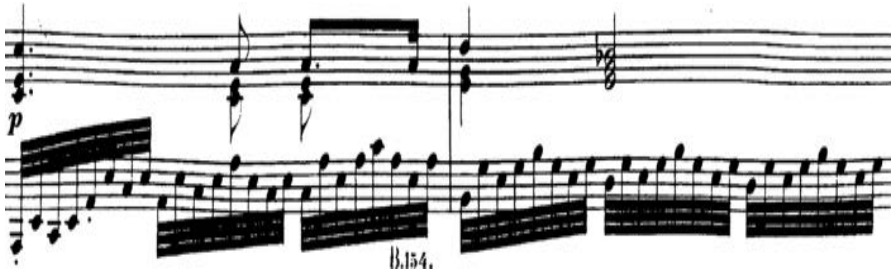
and down another third to Dflat major,



and down another third to Bflat minor,

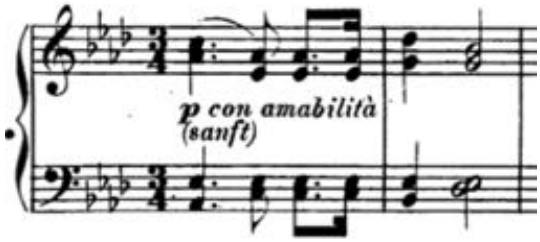


and finally returns back to Aflat major.

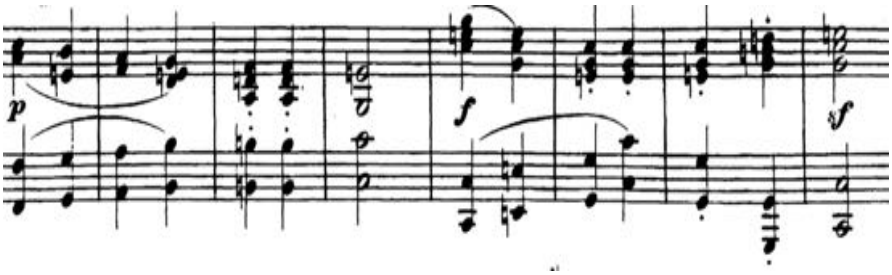


Identifying this sequence of keys,
all appearing in sequence in the short space of 14 measures,
it is a thrill to then realise that the same sequence of keys appears on a much larger scale over the course of the work:

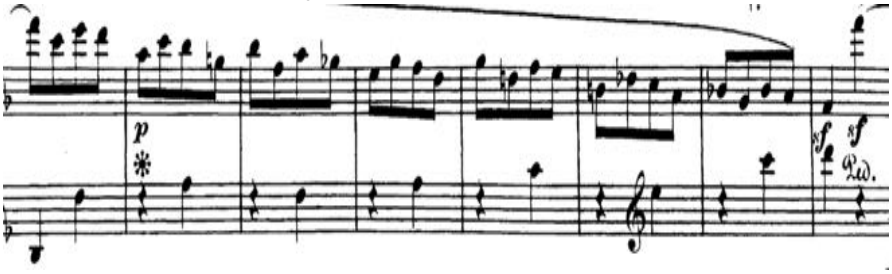
The first movement is in A flat major, (I 1)



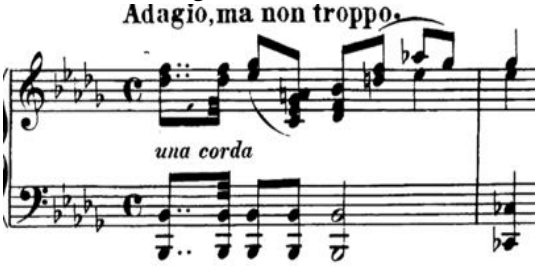
then several minutes later the second movement begins down a third in F minor, (II 1)



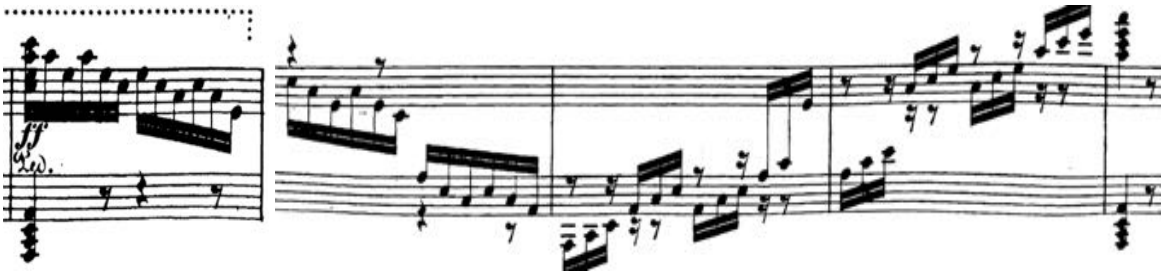
the second part of the second movement, which is the trio,
is down a third in Dflat major, (II 41)



then after the second movement, the beginning of the third movement
is down a third again in Bflat minor (III 1)



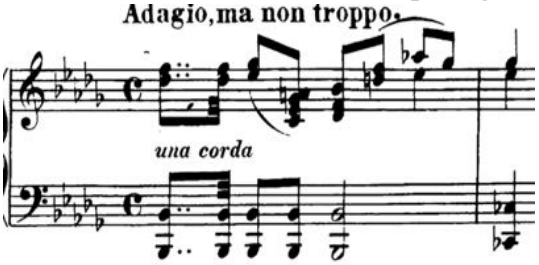
and finally, the work concludes back in the home key of Aflat major.
(III 209)



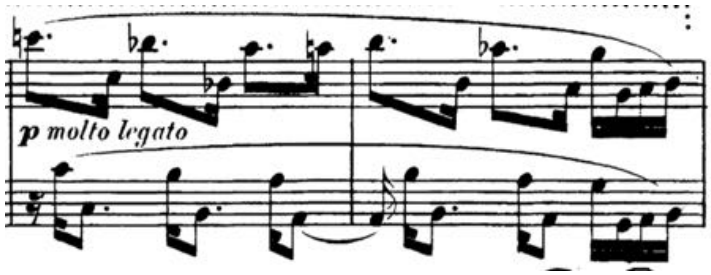
It is truly a grand exploration of scale, expanding the sequence of keys from 16 measures of the "inner planets" to the the span of the whole work and the "outer planets".

The concept of periodicity

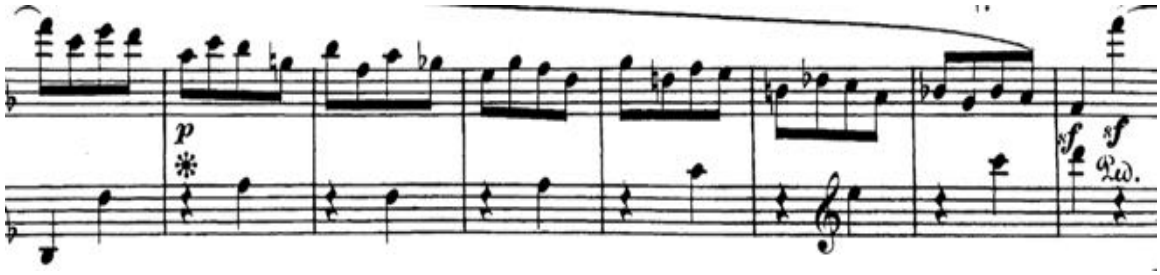
Following these studies of scale, the conversation proceeded to a discussion of phenomena with periodicity.
This is an excerpt from Humboldt's "Personal Narrative" where he has first insights towards the periodicity of the November meteor showers: (III 1 **Opening of 3rd movement**)



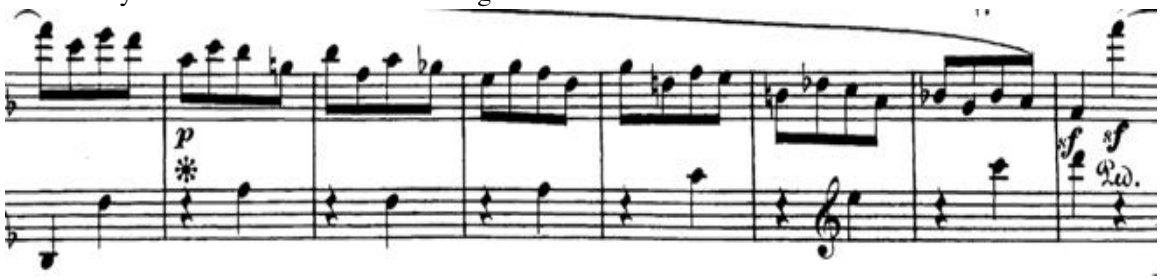
"The night of the 11th of November was cool and extremely fine. (I 20 **white birds circling**)



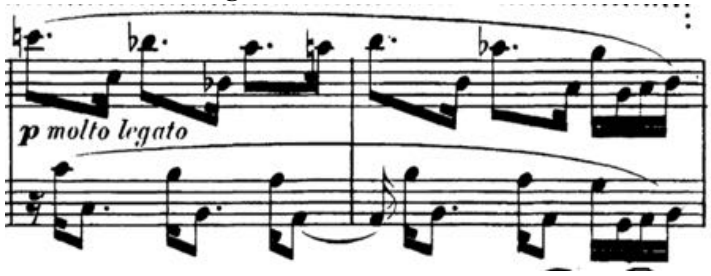
From half after two in the morning, the most extraordinary luminous meteors were seen in the direction of the east.
Thousands of bolides and falling stars succeeded each other during the space of four hours. (II 41 **Trio**)



Some of them attained a height of 40 degrees, and all exceeded 25 or 30 degrees.
M. Bonpland states that, from the first appearance of the phenomenon,
there was not in the firmament a space equal in extent to three diameters of the moon, which was not filled every instant with bolides and falling stars.

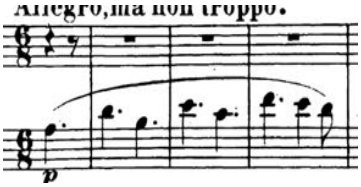


Many of the falling stars had a very distinct nucleus, as large as the disk of Jupiter, from which darted sparks of vivid light.
The bolides seem to burst as by explosion;
but the largest, disappeared without scintillation, leaving behind them phosphorescent bands. (I 20 **white birds circling**)

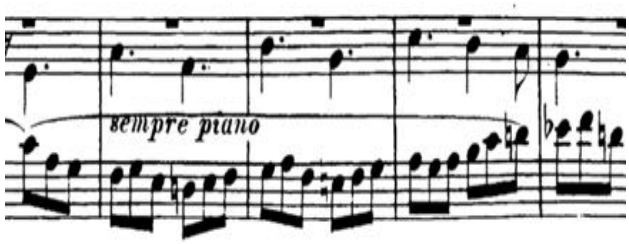


"What was my astonishment, when, on my return to Europe,
I learned that the same phenomenon had been perceived on an extent of the globe of 64 degrees of latitude, and 91 degrees of longitude;
at the equator, in South America, at Labrador, and in Germany!

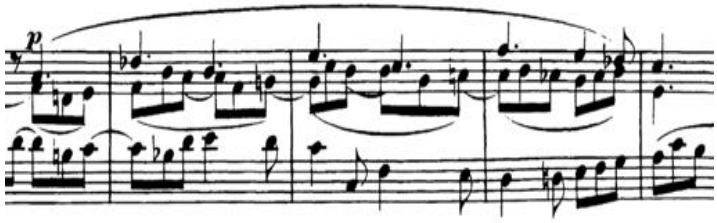
Humboldt definitively recognizes the periodicity of the November meteor showers in the "Kosmos" volumes. (III 27 **first fugue, first entry**)



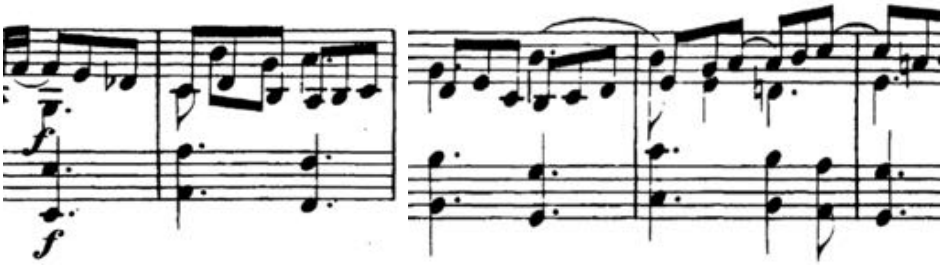
A "great quantity of shooting stars were seen to fall at Potsdam on the night between the 12th and 13th of November, 1822, (III 31 **1st fugue second entry**



and on the same night of the year in 1832 throughout the whole of Europe and even in the southern hemisphere, (III 36 **1st fugue third entry**)

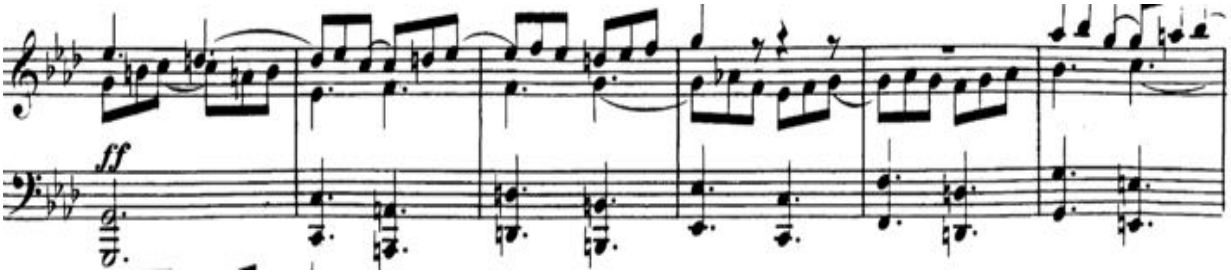


[Notwithstanding this,] no attention was directed to the 'periodicity' of the phenomenon,
until the prodigious swarm of shooting stars which occurred in North America between the 12th and 13th of November, 1833. (III 45 **first fugue 4th entry**)



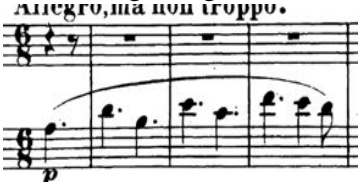
The stars fell on this occasion, like flakes of snow, and it was calculated that at least 240,000 had fallen during a period of nine hours.

[This] was again observed in the United States in 1834,
on the night between the 13th and 14th of November. (III 73 **1st fugue fortissimo Bass octaves**)

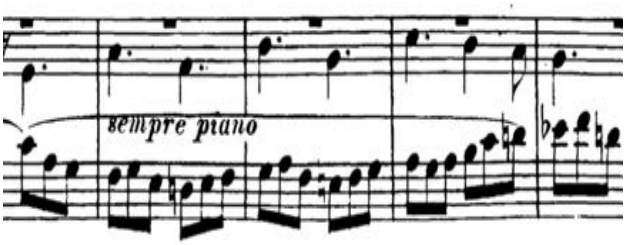


This periodicity was finally explained when the Tempel-Tuttle comet was pinpointed 22 years after the publication of “Kosmos”.
The Tempel-Tuttle comet's great 33 year elliptical orbit around the sun leaves a trail of debris.
Each November 13th the earth's orbit crosses this trail of debris and we see in the skies the Leonid meteor showers.

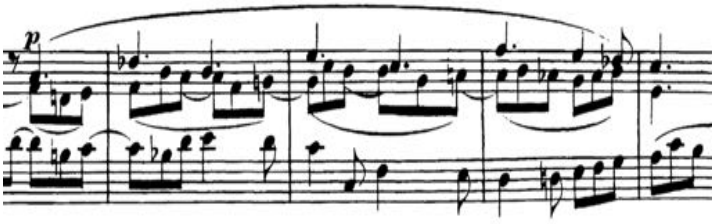
Beethoven's great piano sonata Op. 110 also has a deep explotation of periodicity.
This occurs, most visibly, in the two fugues in the last movement.
The first fugue begins with a short unaccompanied melody or "subject" (III 27 **1st fugue, first entrance**)



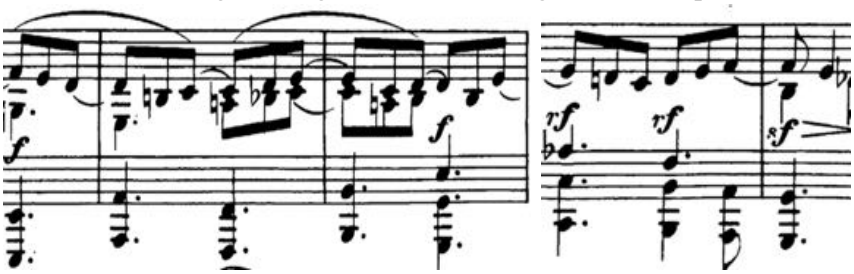
Immediately the next voice enters with the same subject while the first voice continues as an accompaniment. (III 31 **1st fugue, second entrance**)



Now the subject enters in the third voice and is accompanied by the other two. (III 36 **1st fugue, third entrance**)



As the fugues of opus 110 are in 3 voices
the first section of the fugue, the exposition,
has now, theoretically, come to an end.
There are no rules governing the form of the fugue from this point on.



The composer's imagination alone determines the course taken by the rest of the composition.
However, the rules of voice writing are strict,
and the task of writing for freely moving,
independent, voices of equal importance is very challenging indeed.

Circles and Ellipses

After this exploration of periodicity,
the discussion came around to elliptic orbits.
The astronomical researchers of the time
were enthralled by the new understanding of comets and their motion
and Humboldt was discussing with Crelle some of the exciting recent discoveries.

Humboldt: You know Crelle,
I just received a note from Bessel about Dirichlet's article that you published in your journal.
You started the journal two years ago? **NOD**
You seem to have created the top mathematics journal in a flash.
People are now saying that it is as important as The Academy Journal in Paris
and even more important than Schumacher's Astronomische Nachrichten.

Crelle: The credit goes to the excellent articles that I have been able to obtain.

Humboldt: The world is raving about your skill in finding young contributors.
Besides our Dirichlet here, you've published these astounding papers of Abel and Jacobi.
You are only now starting the third volume.
How do you do it?

Crelle: Well, Jacobi is a genius and Abel is a miracle.

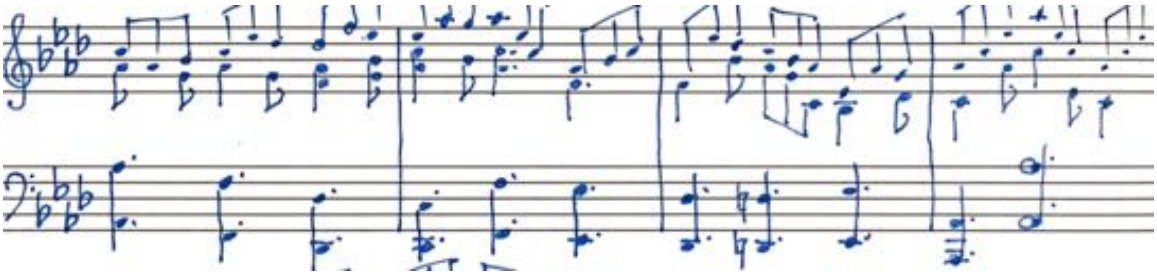
Humboldt: It seems that they have the mathematics for the music of the celestial spheres,
at least that is what I'm being told.

Crelle: Yes indeed, Jacobi and Abel have developed elliptic functions
that generalise our usual sine and cosine circular functions.

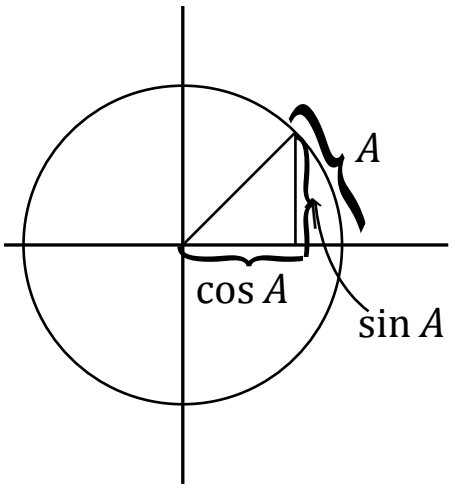
Humboldt: And what is an elliptic function? Can you explain?

Crelle: You first think about the circle. Perfectly round.

Humboldt: You mean like Bach.(**Bach - Jesu joy of man's desiring**)



Crelle: Yes, yes exactly, a perfect circle.
To make it concrete let us take a circle exactly 100mm in diameter.
Suppose you start at one ear, and travel along the circle for exactly 30mm.
What is your position?
The *x*-coordinate is cos(30), in other words, 86mm to the right of center.
and the *y*-coordinate is sin(30), in other words, 50mm above center.



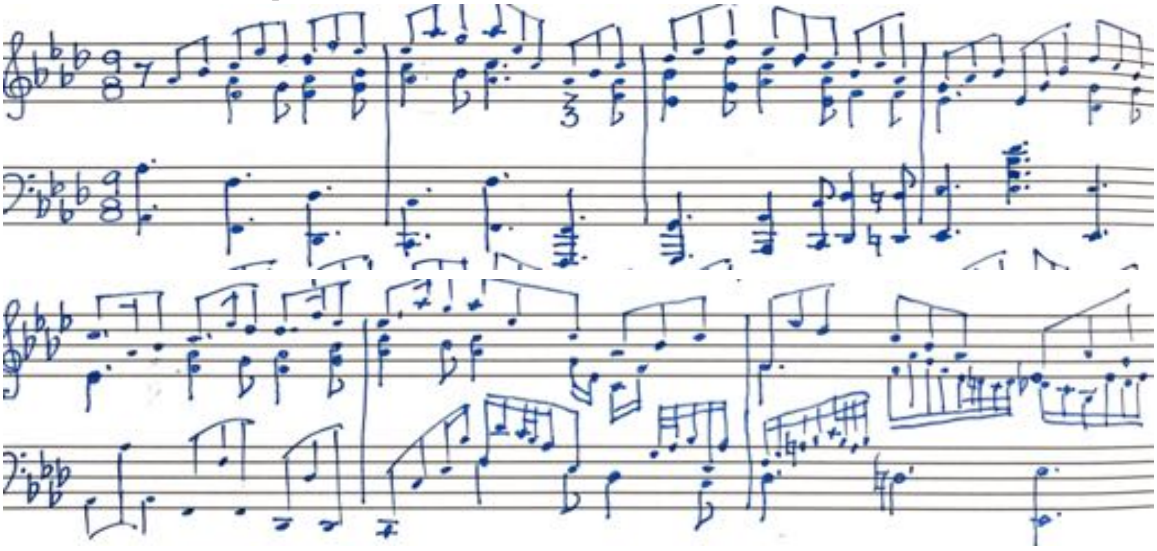
$\sin A$ is the y-coordinate of a point at arclength A on a circle of radius 1

$\cos A$ is the x-coordinate of a point at arclength A on a circle of radius 1

Humboldt: Yes, yes fine, My brother and I learned this when we were boys.
We know everything about circles.

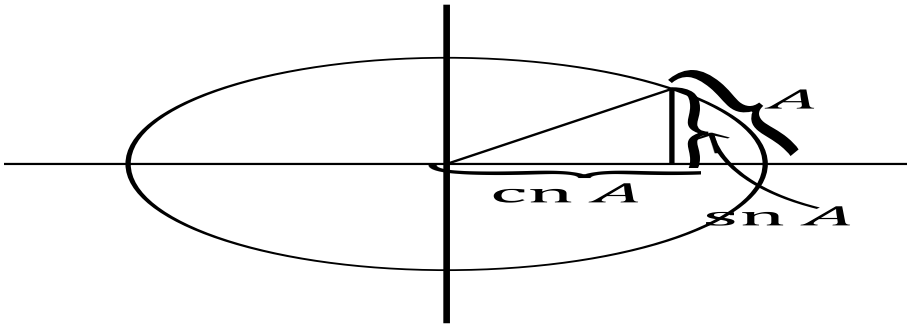
Crelle: The point is that an ellipse is a squashed circle.

Humboldt: You mean squash the Bach to Beethoven! (**Bach to Beethoven**)



Crelle: Bravo! Now that we are at squashed Bach and elliptic Beethvoen,
travel along the ellipse.

Start at one ear, and travel along the ellipse for exactly 30mm.
What is your position on the map?
Then the x-coordinate is elliptic cosine(30), in other words, 56mm to the right of center.
and the y-coordinate is elliptic sine(30), in other words, 8mm to above center.



$\operatorname{sn} A$ is the y-coordinate of a point at arclength A on an ellipse

$\operatorname{cn} A$ is the x-coordinate of a point at arclength A on an ellipse

Humboldt: Fantastic!
This is what is needed to determine exactly the positions of comets.
After all, the comets travel in elliptical orbits!

And if I specify the x and y coordinates
can you tell me the arc lengths along the ellipse?

Crelle: Indeed this is the great insight of Abel and Jacobi,
we have good functions for both determining the arc length from the (x,y) position
and, vice versa, the (x,y) position from the arc length.
These are the inversion formulas.

INVERSION FORMULAS

$$\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}, \qquad \text{and} \qquad \operatorname{sn}^{-1}(x) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

CIRCULAR ELLIPTIC

Humboldt: And are there addition formulas also?, like there are for circular functions?
In other words, if I find the position of the comet today,
and then find the position of the comet tomorrow,
can I then predict where the comet will be in 7 days?

Crelle: Indeed, with the addition formulas for elliptic functions the theory is now complete,
thanks to Abel and Jacobi, published in my journal.

ADDITION FORMULAS

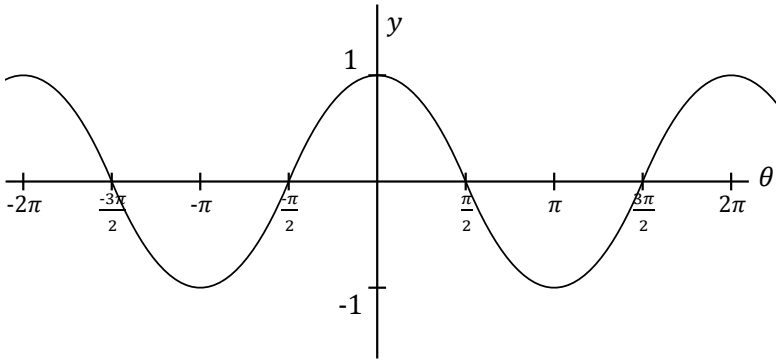
$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \text{and} \qquad \operatorname{sn}(x+y) = \frac{\operatorname{sn} x \operatorname{cn} y \operatorname{dn} y + \operatorname{sn} y \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}$$

CIRCULAR ELLIPTIC

Humboldt: It perfectly describes exactly the positions and motion of comets,
isn't that what we mean by the music of the celestial spheres?

But remember,
the graph of the sine and cosine, the circular functions, they are periodic,
they have this perfect periodicity as waves,
in fact it was Galileo's father that told us that this is exactly the motion of the vibrating string in my piano.

$y = \cos\theta$



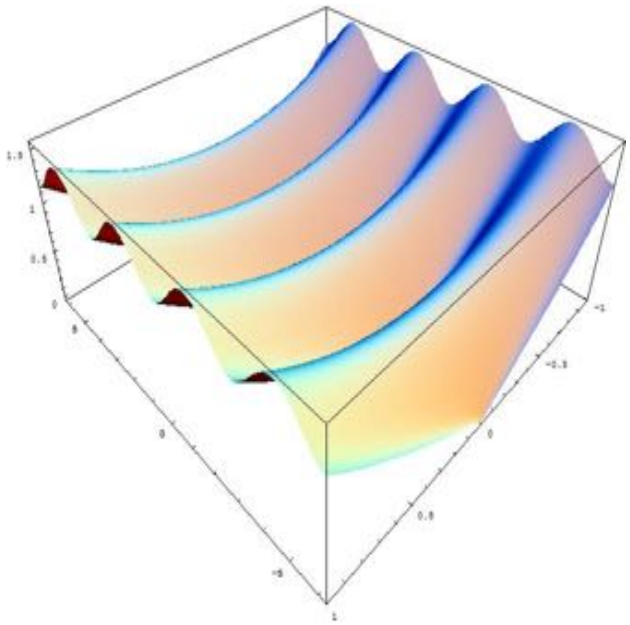
This periodicity is like the periodicity of the November meteor showers
or the periodicity of the subject in a fugue.
Is there anything like that periodicity in the elliptic functions?

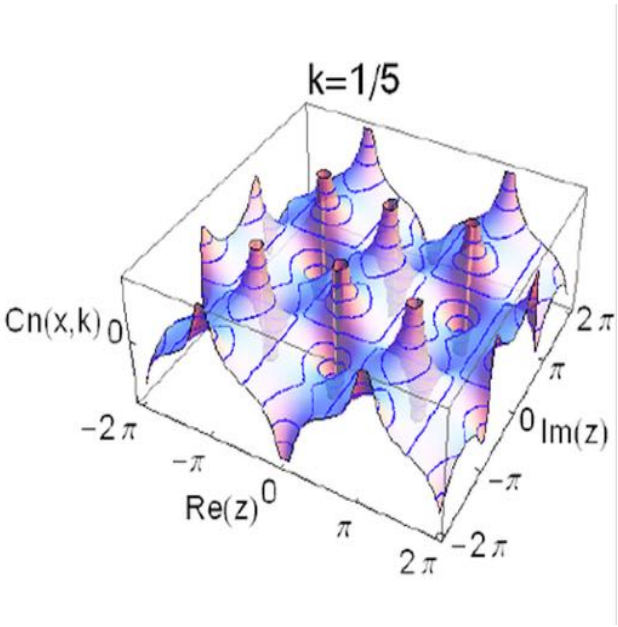
Crelle: Yes, this is, in fact, the most amazing aspect, of all.
But to see it best one should think about complex numbers.
This is what Gauss has made clear to us, following Euler.

Euler's formula is already using complex numbers for circular functions.

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

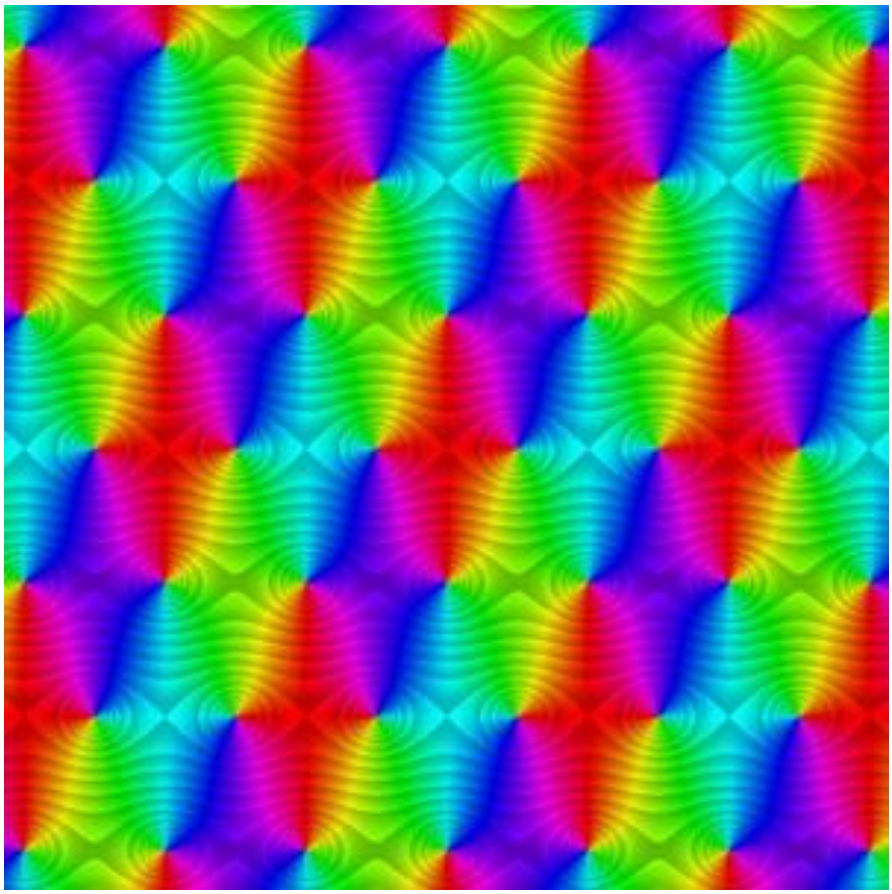
If we view elliptic functions as functions in complex numbers,
we find that there is a DOUBLE periodicity,





CIRCULAR ELLIPTIC

in two directions, both the real direction and the imaginary direction!

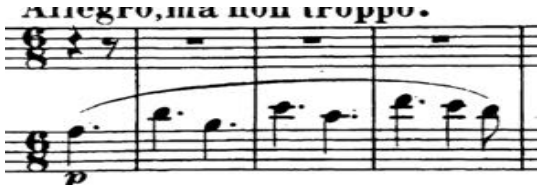


Humboldt:
If that is not as beautiful as music then what is?

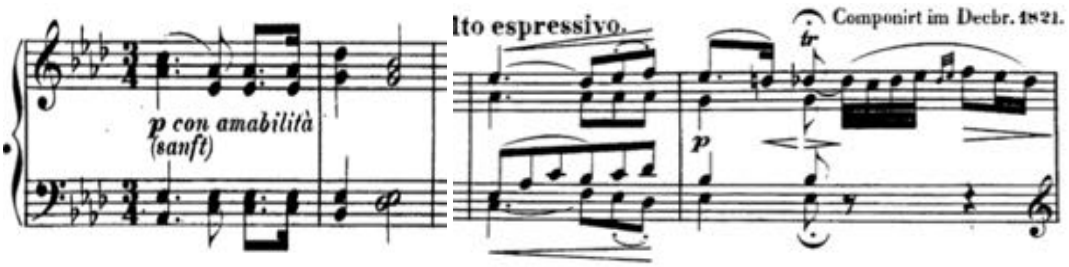
At this point Abraham Mendelssohn called together all the guests:

Abraham Mendelssohn: I think that now it is a good moment to hear some music.
Prof. Zelter here will play for us the Beethoven piano sonata Op. 110.
It must be one of Beethoven's last works,
our great Beethoven died only last year.
Felix gave Fanny the music to Op 106 on her birthday a few years ago (**GESTURE IN AWE**)
and the Op. 110 is an even later work. Is it very modern?

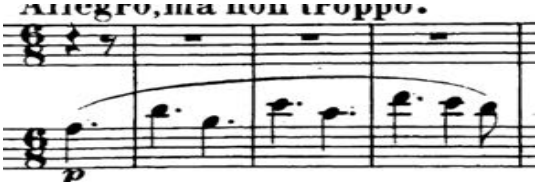
Zelter: Oh yes, and as with all his late works I believe this sonata
will never not be modern and will always astonish.
Beethoven is a magician.
The fugue theme of the last movement



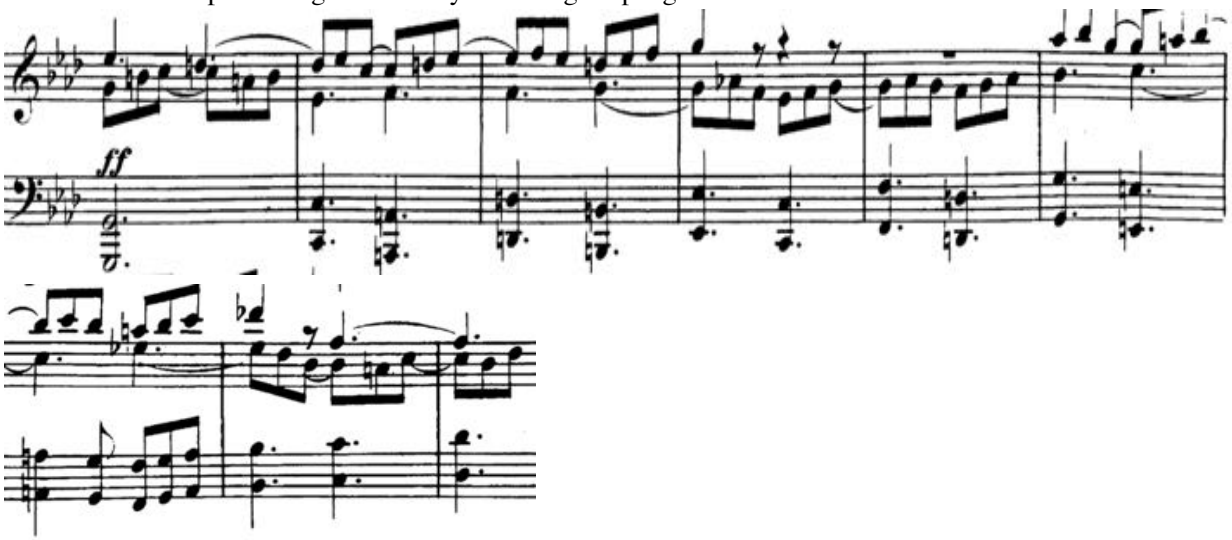
is distilled from the very opening of the sonata:



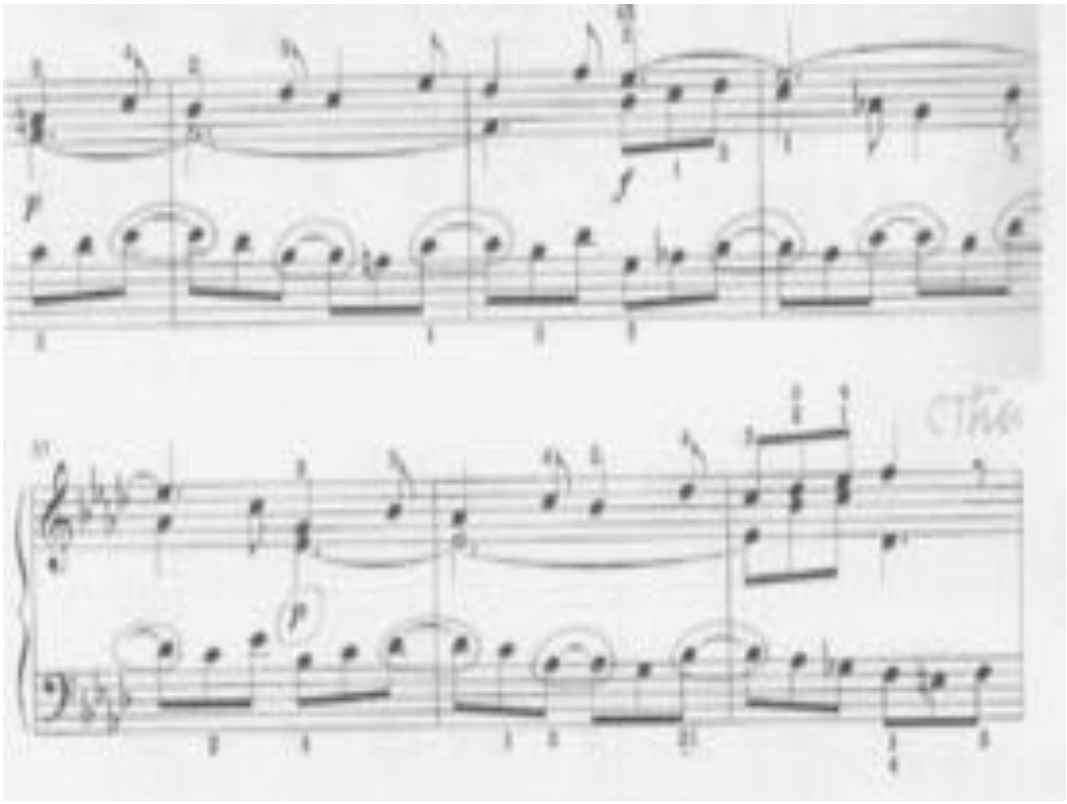
This is the fugue theme



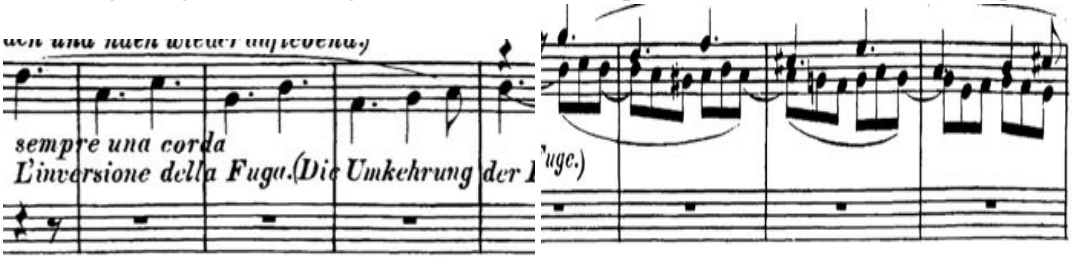
Beethoven develops this fugue theme by marching it up a giant staircase:



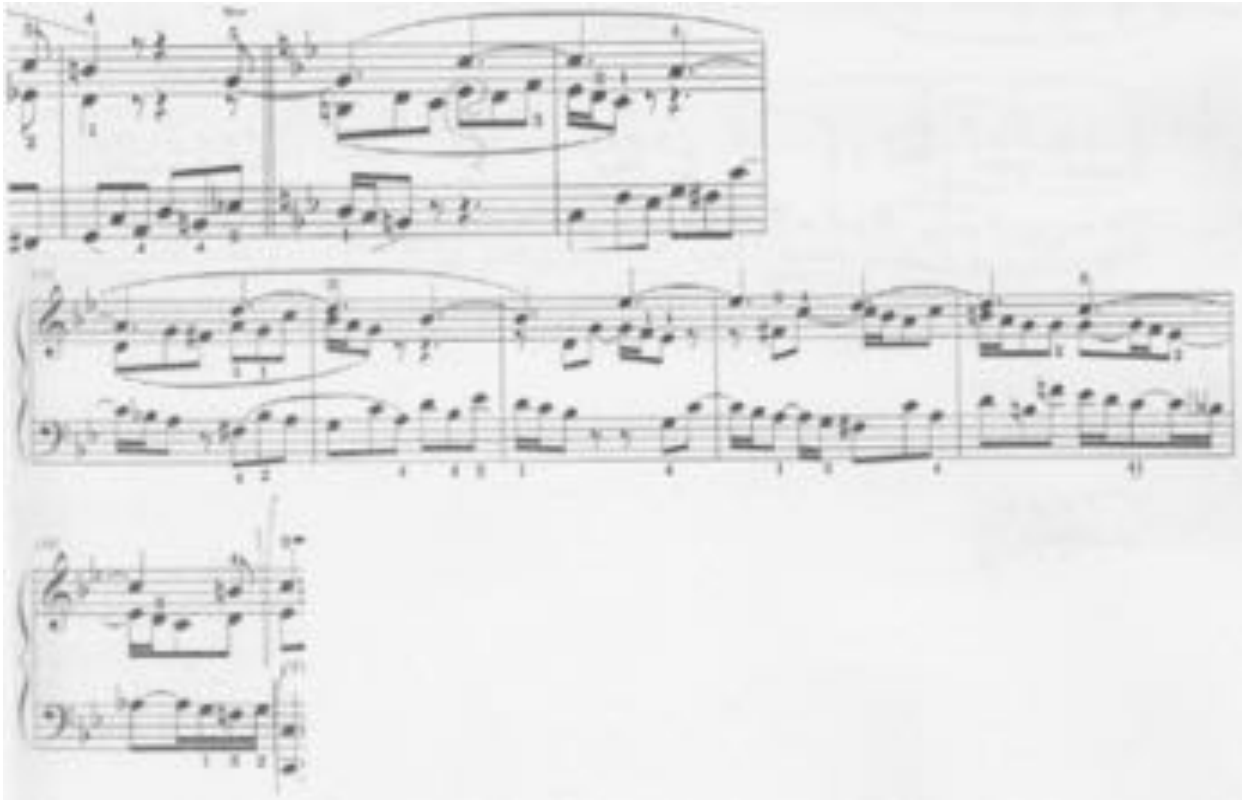
At the top the theme relaxes:



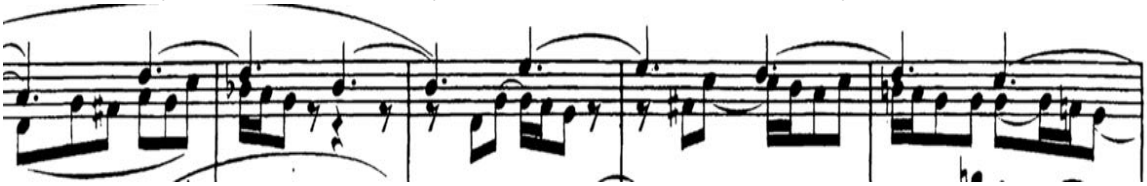
At the beginning of the 2nd fugue the theme floats in upside down like an enormous soap bubble.



Eight bars later it appears fast and intent like 2 ferret in a rabbit hole.

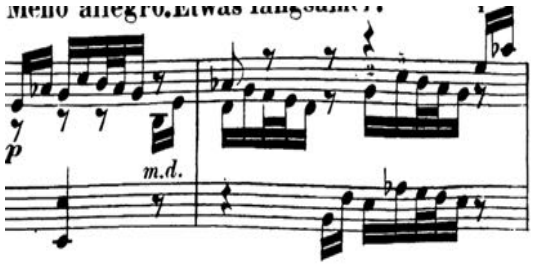


simultaneously it is stretched out lazily lik a white cloud on a summers day.



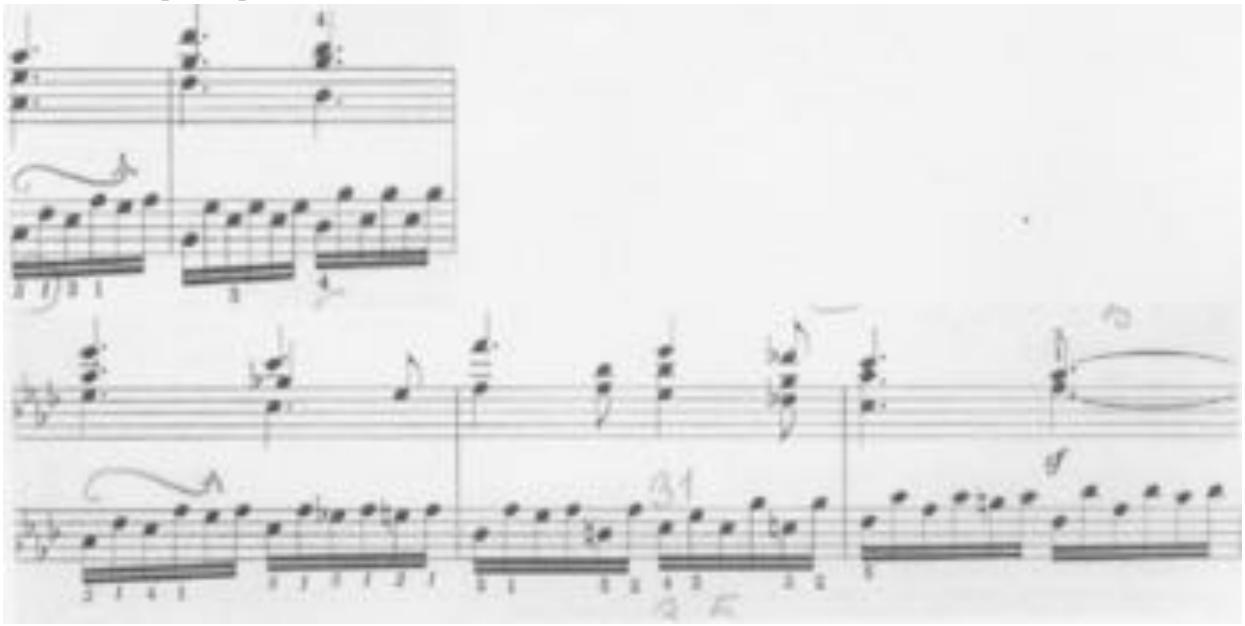
Here is the unlikely meeting of the cloud and the ferrets.

Delete 2 notes of the theme and speed it up 6 times to produce the electric eel music.



Finally the theme is reduced to a tiny quivering flourish:

which flares up at unpredictable intervals.



This is not an empty display of virtuosity in fugal writing.
By this astonishing process Beethoven creates a huge upsurge of energy end tension.

Like Humboldt, Beethoven was an explorer,
a solitary figure on a high peak,
where the air is thin, the view is spectacular,
and the cosmos is within the grasp of his outstretched hand.

Thank you.

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- The three dimensional image of the complex circular function is https://i.stack.imgur.com/HKKGH.jpg from https://math.stackexchange.com/questions/2050426/how-can-i-show-the-complex-sine-function-is-unbounded
- The three dimensional image of the complex elliptic function is https://www.researchgate.net/profile/Carl_Bender/publication/45892999/figure/fig12/AS:307331800813569@1450285019379/FIG-12-Real-part-of-the-cnoidal-elliptic-function-Cn-x-k-in-the-complex-x-plane.png from https://www.researchgate.net/figure/45892999_fig12_FIG-12-Real-part-of-the-cnoidal-elliptic-function-Cn-x-k-in-the-complex-x-plane
- The two dimensional brightly colored image of the complex elliptic function is http://www.mathe.tu-freiberg.de/~wegert/PhasePlot/EllipticCn.png from http://www.visual.wegert.com/

The following resources were fundamental in the research process for this lecture.

- L. van Beethoven, Complete Score to *Piano Sonata No. 31, Op. 110*, Edited by Johannes Brahms, Breitkopf und Härtel, Leipzig 1862. available from IMSLP/Petrucci Music Library http://imslp.org/wiki/Piano_Sonata_No.31,_Op.110_(Beethoven,_Ludwig_van).
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