International Conference on Schubert Calculus

Affine flag varieties Guangzhou, China 7 Nov. 2017 (1)

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~~$SL_{n+1}[\epsilon,\epsilon^{-1}] = \{(a_{ij})$~~ $G = SL_{n+1}(\mathbb{C}[\epsilon,\epsilon^{-1}])$

$$I^+ = \{(g_{ij}) \in SL_{n+1}(\mathbb{C}[\epsilon]) \mid (g_{ij}(0)) \in \{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\}\}$$

$$I^0 = \{(g_{ij}) \in SL_{n+1}(\mathbb{C}[\epsilon,\epsilon^{-1}]) \mid (g_{ij}) \in \{\begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}\}\}$$

$$I^- = \{(g_{ij}) \in SL_{n+1}(\mathbb{C}[\epsilon^{-1}]) \mid (g_{ij}(\infty)) \in \{\begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}\}\}$$

G/I^+ pos. level (thin)

G/I^0 level 0 affine flag varieties (semiinfinite)

G/I^- neg level (thick)

W = affine Weyl group = {alcoves}

$$G = \bigsqcup_{x\in W} I^+ x I^+ = \bigsqcup_{y\in W} I^+ y I^0 = \bigsqcup_{z \in W} I^+ z I^-$$

Define

$x \le w$ if $I^+ x I^+ \subseteq \overline{I^+ w I^+}$

$x \preceq w$ if $I^+ x I^0 \subseteq \overline{I^+ w I^0}$

$x \ge w$ if $I^+ x I^- \subseteq \overline{I^+ w I^-}$

Pieri-Chevalley: In $K_{I^+ \rtimes \mathbb{C}^*}(G/I^0)$

$$[\mathcal{L}(\lambda + 0\Lambda_0)][\mathcal{O}_{\overline{I^+ x I^0}}] = \sum_{\substack{p \in B(\lambda+0\Lambda_0) \\ \iota(p) \succeq x}} e^{\mathrm{end}(p)} [\mathcal{O}_{\overline{I^+ \varphi(p) I^0}}]$$

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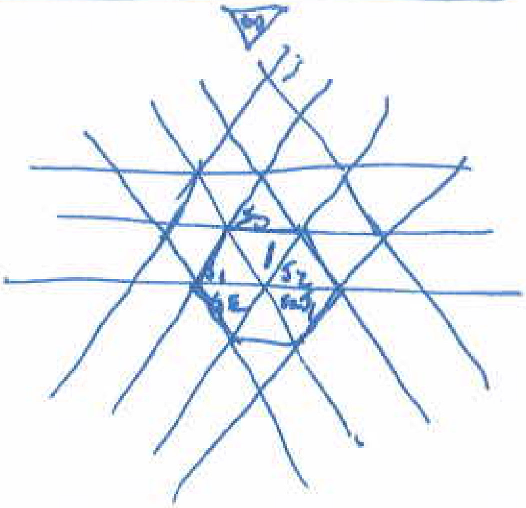
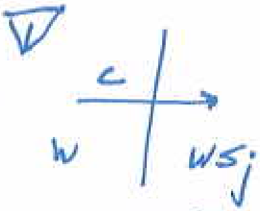
Labeled alcove walks



Alcove walks start at 1

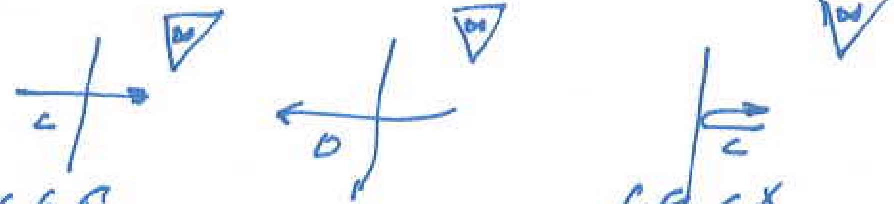
For each $w \in W$ fix
$\vec{w} = s_{i_1} \cdots s_{i_\ell}$
a reduced word.

Blue step type j



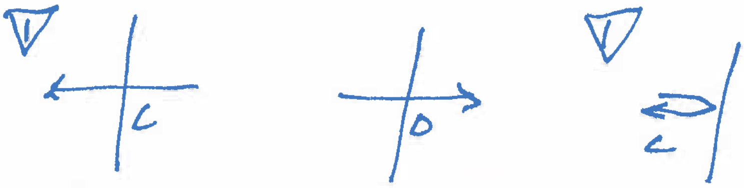
$c \in \mathbb{C}$

Green step type j



$c \in \mathbb{C}$ $\qquad$ $c \in \mathbb{C}^\times$

Red step type j



$c \in \mathbb{C}$ $\qquad$ $c \in \mathbb{C}^\times$

$$I^+ x I^+ = \left\{ \begin{matrix}\text{blue walks type } \vec{w} \\ \text{that end in } x\end{matrix} \;\middle|\; w \in W \right\}$$

$$I^+ y I^0 = \left\{ \begin{matrix}\text{green walks type } \vec{w} \\ \text{that end in } y\end{matrix} \;\middle|\; w \in W \right\}$$

$$I^+ z I^- = \left\{ \begin{matrix}\text{red walks type } \vec{w} \\ \text{that end in } z\end{matrix} \;\middle|\; w \in W \right\}.$$

Affine Lie algebras / Quantum groups

$$\mathfrak{sl}_{n+1}[\epsilon,\epsilon^{-1}] = \left\{ (a_{ij}) \;\middle|\; \begin{array}{l} a_{ij} \in \mathbb{C}[\epsilon,\epsilon^{-1}] \\ a_{11}+\cdots+a_{nn}=0 \end{array} \right\}$$

with $[A,B] = AB - BA$.

$$\mathfrak{g} = \mathfrak{sl}_{n+1}[\epsilon,\epsilon^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with $[K,d]=0$, $[K,A\epsilon^m]=0$, $[d,A\epsilon^m] = mA\epsilon^m$

$$[A\epsilon^m, B\epsilon^s] = (AB-BA)\epsilon^{m+s} + \delta_{m,-s}\, tr(AB)\cdot K$$

where $A \in \mathfrak{sl}_{n+1}$. Then $\mathfrak{g}$ is Kac-Moody!

Generators and Serre relations

$\mathfrak{g}$ $\quad e_0, e_1, \dots, e_n, f_0, f_1, \dots f_n, h_1, \dots, h_n, K, d$

$\cup\!|$

$\mathfrak{b}$ $\quad e_0, e_1, \dots e_n, \quad h_1, \dots, h_n, K, d$

$\cup\!|$

$\mathfrak{h}$ $\quad h_1, \dots, h_n, K, d$

$\cup\!|$

$\mathfrak{a}$ $\quad h_1, \dots, h_n.$

Let $(\mathfrak{sl}_2)_i := \operatorname{span}\{e_i, f_i, h_i\}$ with $h_i = [e_i, f_i]$

A $\mathfrak{g}$-module M is integrable if

$$\operatorname{Res}^{\mathfrak{g}}_{(\mathfrak{sl}_2)_i}(M) = \bigoplus \begin{pmatrix} \text{fin.-dim'l} \\ (\mathfrak{sl}_2)_i\text{-modules} \end{pmatrix}, \quad i \in \{0,\dots,n\}.$$

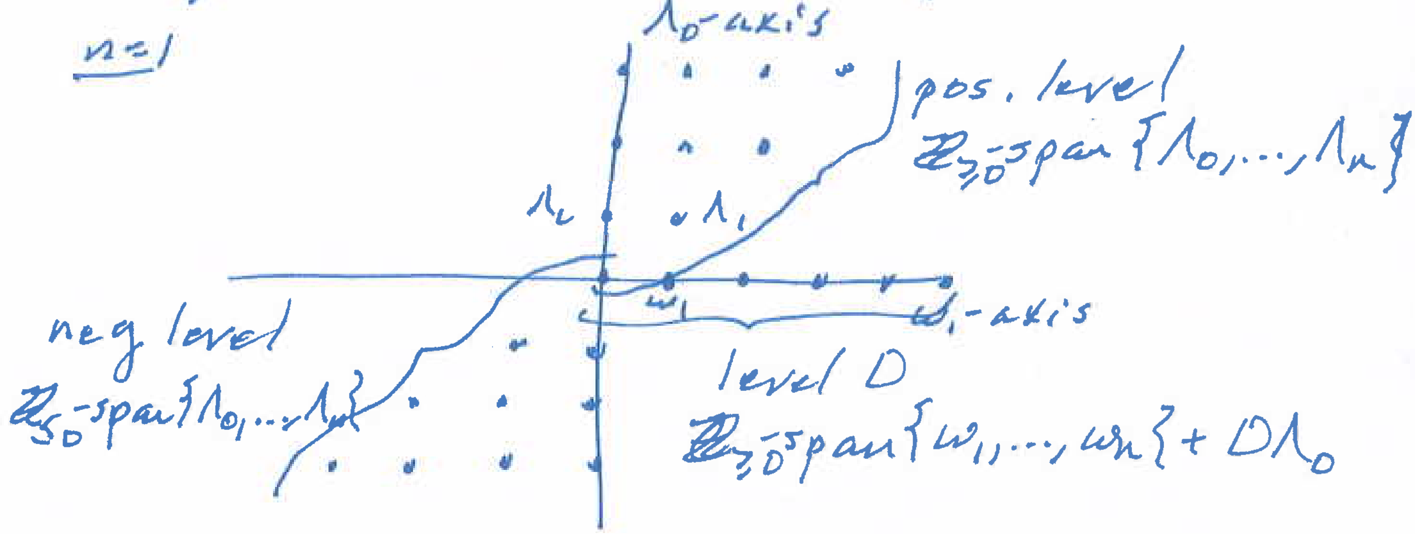
W acts on $\mathfrak{h}^*$ and on M.

Indexing simple modules

simple $\mathfrak{a}$-modules $\quad \lambda \in \mathfrak{a}^* = \mathbb{C}\omega_1 + \cdots + \mathbb{C}\omega_n$

simple $\mathfrak{h}$-modules $\quad \Lambda \in \mathfrak{h}^* = \mathbb{C}\omega_1 + \cdots + \mathbb{C}\omega_n + \mathbb{C}\Lambda_0 + \mathbb{C}\delta.$

integrable $\mathfrak{g}$-modules $\quad \Lambda \in (\mathfrak{h}^*)_{int}$



Let $\Lambda \in (\mathfrak{h}^*)_{int}$. The <u>extremal weight module</u> $L(\Lambda)$ is the $\mathfrak{g}$-module gen by $\{u_{w\Lambda} \mid w \in W\}$

$$h_i u_{w\Lambda} = \langle w\Lambda, \alpha_i^\vee \rangle u_{w\Lambda}$$

$e_i u_{w\Lambda} = 0$ and $f_i^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}$ if $\langle w\Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$

$e_i^{-\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}$ and $f_i u_{w\Lambda} = 0$ if $\langle w\Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\leq 0}$.

Let $x \in W$. The <u>Demazure module</u>

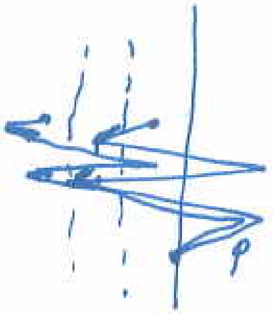
$$L(\Lambda)_{\geq x} = (U\mathfrak{b})\, u_{x\Lambda}.$$

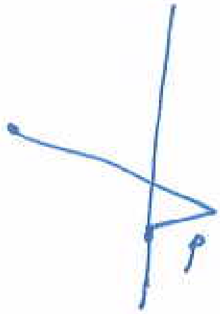
<u>Borel-Bott-Weil</u> Let $\Lambda = \lambda + 0\Lambda_0 \in (\mathfrak{h}^*)_{int}$ and $x \in W$.

$$H^0(\overline{I^+ x I^0}; \mathcal{L}(\Lambda)) \cong L(\Lambda)_{\geq x}.$$

Crystals of extremal weight modules

$\tilde{f}_i p$

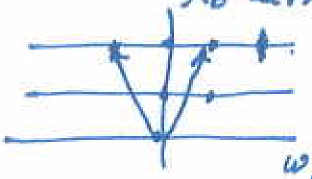


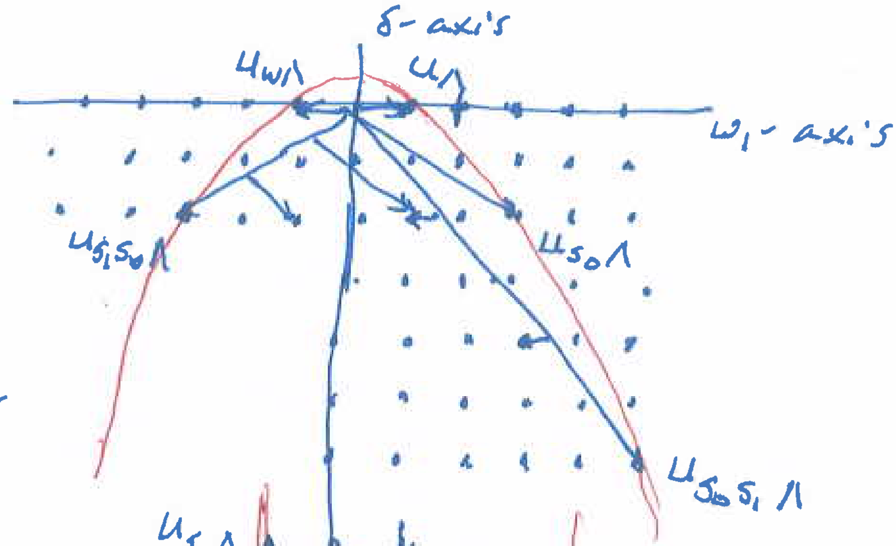


$\tilde{f}_i p = 0$

Positive level

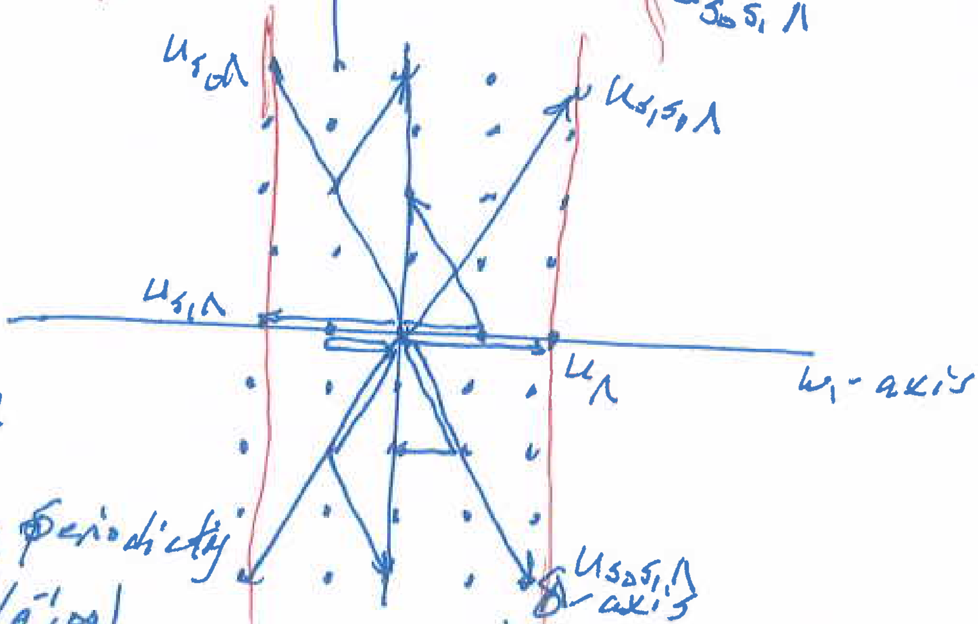
$\Lambda = \omega_1 + 2\Lambda_0$





Level 0

$\Lambda = 2\omega_1 + 0\Lambda_0$



Periodicity and

Character Ignoring periodicity

$ch(L(\Lambda)) = E_{w_0\Lambda}(q^{-1}, \infty)$

$\leftarrow$ nonsymmetric Macdonald polynomial.

Negative level

$\Lambda = -\omega_1 - 2\Lambda_0$