Combinatorics of level 0 representations

Affine flag varieties

Future directions in Representation Theory
University of Sydney 4-8 Dec. 2017

Loop groups

$$G = SL_{n+1}(\mathbb{C}[\epsilon,\epsilon^{-1}]) = \{(g_{ij}) \mid g_{ij} \in \mathbb{C}[\epsilon,\epsilon^{-1}],\ \det(g_{ij}) = 1\}$$

Borel subgroups

$$I^+ = \{(g_{ij}) \in SL_{n+1}(\mathbb{C}[\epsilon]) \mid (g_{ij}(0)) \in \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\}$$

$$I^0 = \{(g_{ij}) \in SL_{n+1}(\mathbb{C}[\epsilon,\epsilon^{-1}]) \mid g_{ij} \in \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}\}$$

$$I^- = \{(g_{ij}) \in SL_{n+1}(\mathbb{C}[\epsilon^{-1}]) \mid (g_{ij}(\infty)) \in \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}\}$$

Flag varieties

G/I^+ pos. level aff. flag var. (thin)

G/I^0 level 0 aff flag var. (semiinfinite)

G/I^- neg level aff. flag var. (thick)

Double cosets

$W = \{\text{alcoves}\}$ affine Weyl group

indexes double Cosets:

$$G = \bigsqcup_{x\in W} I^+ x I^+ = \bigsqcup_{y\in W} I^+ y I^0 = \bigsqcup_{z\in W} I^+ z I^-$$

Schubert varieties

Define

$x \le w$ if $I^+ x I^+ \subseteq \overline{I^+ w I^+}$ pos. level Bruhat order

$x \precsim w$ if $I^+ x I^0 \subseteq \overline{I^+ w I^0}$ level 0 Bruhat order

$x \geq w$ if $I^+ x I^- \subseteq \overline{I^+ w I^-}$ neg. level Bruhat order

Pieri-Chevalley formula

In $K_{I^+ \times \mathbb{C}^\times}(G/I^0)$,

$$[\mathcal{L}(\lambda + 0\Lambda_0)][\mathcal{O}_{\overline{I^+ w I^0}}] = \sum_{\substack{p\in B(\lambda+0\Lambda_0)\\ \mathrm{init}(p)}} e^{\mathrm{end}(p)} [\mathcal{O}_{\overline{I^+ \varphi(p) I^0}}].$$

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Integrable $\mathfrak{g}$-modules

$$\mathfrak{sl}_{n+1} = \{ A = (a_{ij}) \mid a_{ij} \in \mathbb{C},\ \mathrm{tr}(A) = 0 \}$$

with $[A,B] = AB - BA$. Then

$$\mathfrak{g} = \Big(\bigoplus_{k \in \mathbb{Z}} \mathfrak{sl}_{n+1} t^k \Big) \oplus \mathbb{C}K \oplus \mathbb{C}d \quad \text{with}$$

$$[K, At^k] = 0, \quad [K, d] = 0, \quad [d, At^k] = k\, At^k$$

$$[At^k, Bt^l] = (AB - BA) t^{k+l} + \delta_{k,-l}\, \mathrm{tr}(AB) \cdot K$$

for $A, B \in \mathfrak{sl}_{n+1}$. Then $\mathfrak{g}$ is Kac-Moody! presented by generators and Serre relations

$$\begin{array}{ll}
\mathfrak{g} & e_0, e_1, \ldots, e_n, f_0, f_1, \ldots, f_n\ \ h_1, \ldots, h_n, K, d \\
\cup\!\mid & \\
\mathfrak{b} & e_0, e_1, \ldots, e_n \qquad\qquad\qquad\ \ h_1, \ldots, h_n, K, d \\
\cup\!\mid & \\
\mathfrak{h} & \qquad\qquad\qquad\qquad\qquad\qquad\ \ h_1, \ldots, h_n, K, d \\
\cup\!\mid & \\
\mathfrak{a} & \qquad\qquad\qquad\qquad\qquad\qquad\ \ h_1, \ldots, h_n
\end{array}$$

Let $(\mathfrak{sl}_2)_i := \mathrm{span}\{e_i, f_i, h_i\}$ with $h_i = [e_i, f_i]$.

A $\mathfrak{g}$-module M is <u>integrable</u> if

$$\mathrm{Res}^{\mathfrak{g}}_{(\mathfrak{sl}_2)_i}(M) = \bigoplus \big(\text{fin. dim'l } (\mathfrak{sl}_2)_i\text{-modules}\big)$$

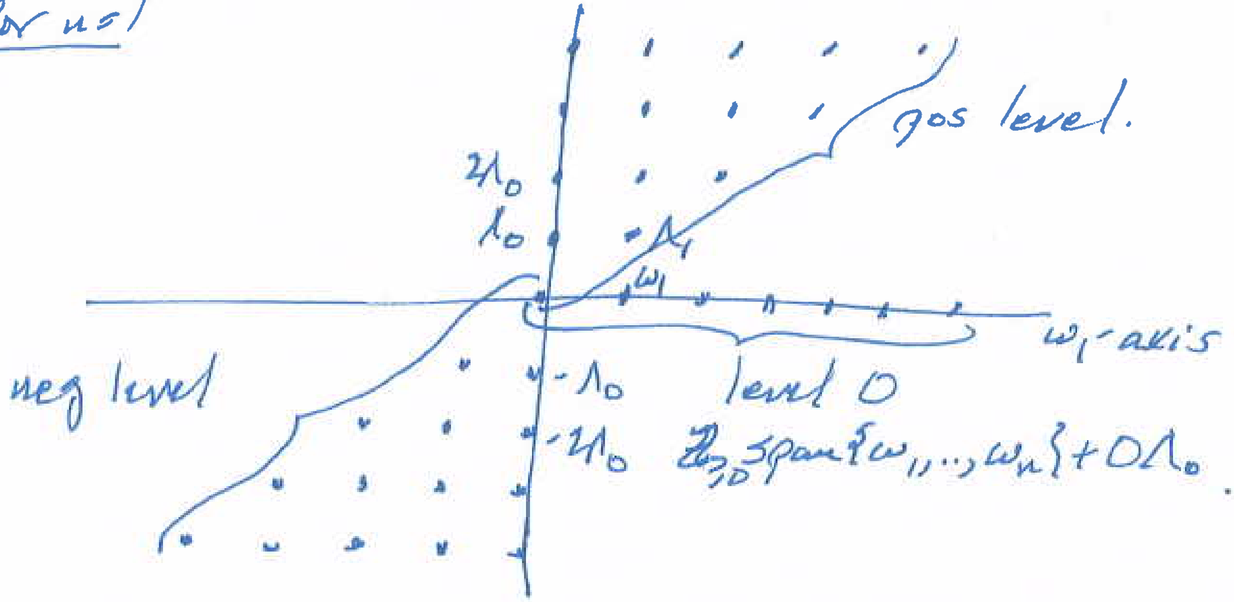
Extremal weight modules and Demazure modules

simple $\mathfrak{a}$-modules $\qquad \lambda \in \mathfrak{a}^+ = \mathrm{span}\{\omega_1,\dots,\omega_n\}$

simple $\mathfrak{g}$-modules $\qquad \lambda \in \mathfrak{h}^* = \mathrm{span}\{\omega_1,\dots,\omega_n,\Lambda_0,\delta\}$.

integrable $\mathfrak{g}$-modules $\qquad \lambda \in (\mathfrak{h}^*)_{int}$

For $n=1$



Let $\Lambda \in (\mathfrak{h}^*)^{int}$. The extremal weight module $L(\Lambda)$ is generated by $\{u_{w\Lambda} \mid w \in W\}$ with

$$h_i u_{w\Lambda} = \langle w\Lambda, \alpha_i^\vee\rangle u_{w\Lambda}.$$

$e_i u_{w\Lambda} = 0$ and $f_i^{\langle w\Lambda,\alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}$ if $\langle w\Lambda,\alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$.

$e_i^{\langle w\Lambda,\alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}$ and $f_i u_{w\Lambda} = 0$, if $\langle w\Lambda,\alpha_i^\vee\rangle \in \mathbb{Z}_{\leq 0}$.

Let $w \in W$. The Demazure module is

$$L(\Lambda)_{\geq w} = (U\mathfrak{b})\, u_{w\Lambda}.$$

Borel-Bott-Weil Let $\lambda+\ell\Lambda_0 \in (\mathfrak{h}^*)^{int}$ and $w \in W$.

$$L(\lambda+\ell\Lambda_0) \simeq H^0(\mathcal{O}_{\overline{I^+ w I^0}}, \mathcal{L}(\lambda+\ell\Lambda_0)).$$

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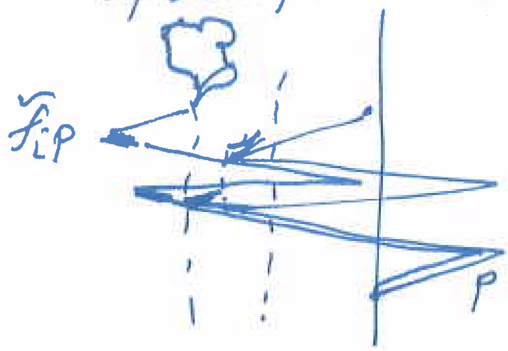
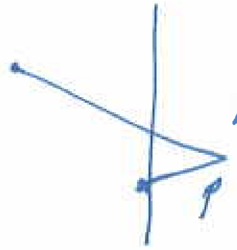
Crystals $B(\Lambda)$

Pathsp and root operators $\tilde{f}_i$



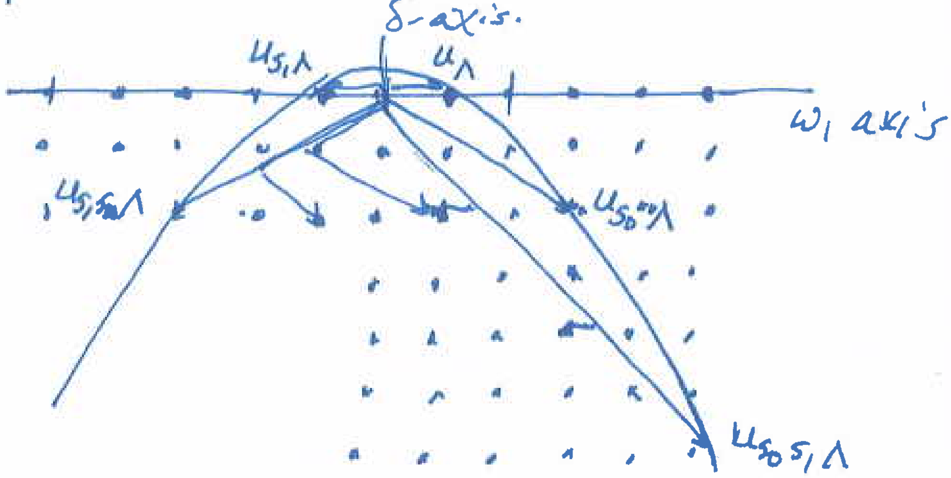
and



has $\tilde{f}_i p$.

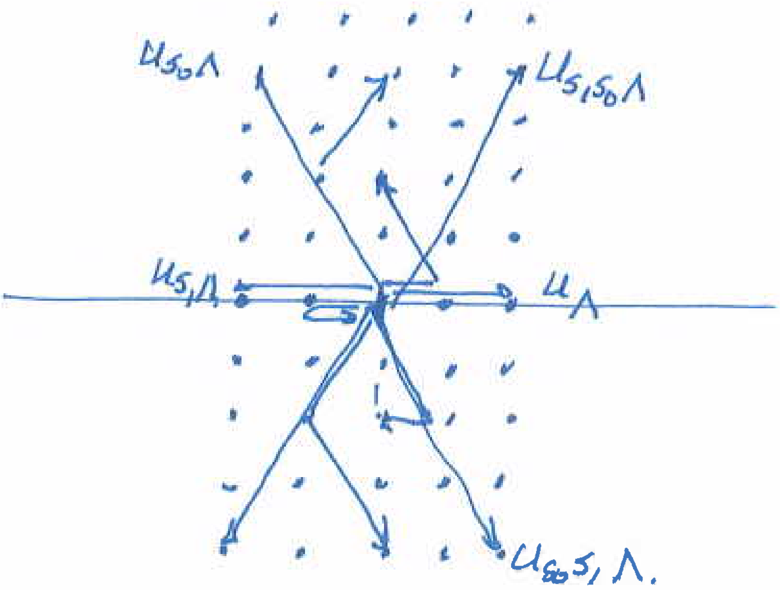
level 2

$\Lambda = \omega_1 + 2\Lambda_0$



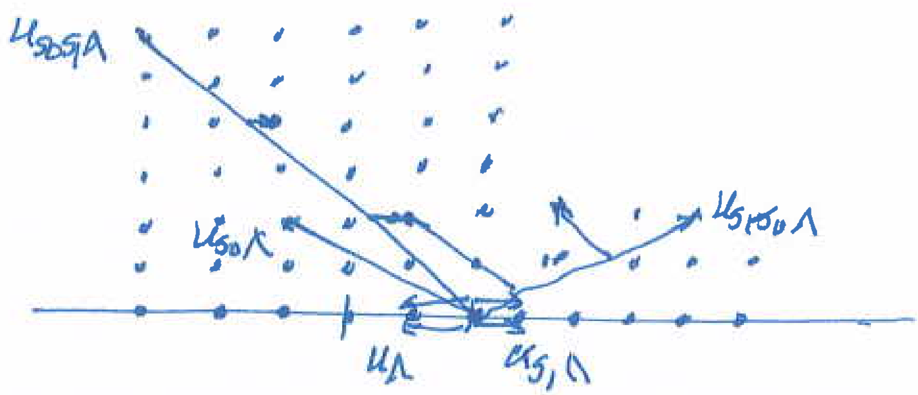
level 0

$\Lambda = 2\omega_1 + 0\Lambda_0$



level -2

$\Lambda = \omega_1 - 2\Lambda_0$



Affine flag varieties

$$G = SL_{n+1}(\mathbb{C}[\epsilon,\epsilon^{-1}]) = \left\{ (g_{ij}) \;\middle|\; \begin{array}{l} g_{ij} \in \mathbb{C}[\epsilon,\epsilon^{-1}] \\ \det(g_{ij}) = 1 \end{array} \right\}$$

$$I^+ = \left\{ g \in SL_{n+1}(\mathbb{C}[\epsilon]) \;\middle|\; (g_{ij}(0)) \in \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \right\}$$

$$I^0 = \left\{ g \in SL_{n+1}(\mathbb{C}[\epsilon,\epsilon^{-1}]) \;\middle|\; g_{ij} \in \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \right\}$$

$$I^- = \left\{ g \in SL_{n+1}(\mathbb{C}[\epsilon^{-1}]) \;\middle|\; (g_{ij}(\infty)) \in \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \right\}$$

G/I^+ positive level aff. flag var. (thin)

G/I^0 level 0 aff. flag var. (semiinfinite)

G/I^- negative level aff flag var. (thick)

W = affine Weyl group, indexes double cosets

$$G = \bigsqcup_{x \in W} I^+ x I^+ = \bigsqcup_{y \in W} I^+ y I^0 = \bigsqcup_{z \in W} I^+ z I^-$$

Define

$x \geq w$ if $I^+ x I^+ \subseteq \overline{I^+ w I^+}$ pos. level Bruhat order

$x \geq\!\!\!\!/\; w$ if $I^+ x I^0 \subseteq \overline{I^+ w I^0}$ level 0 Bruhat order

$x \leq w$ if $I^+ x I^- \subseteq \overline{I^+ w I^-}$ neg. level Bruhat order.