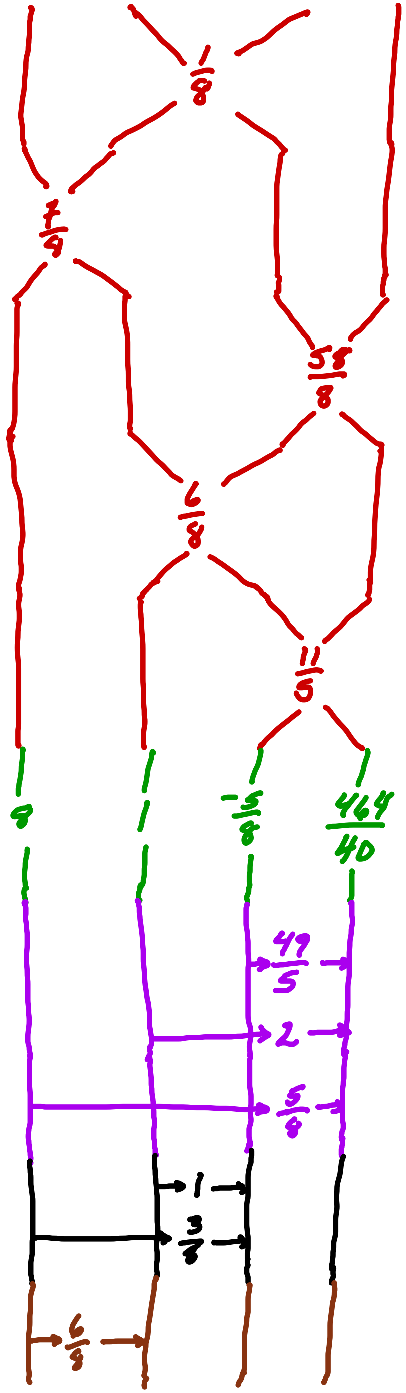
NZMRI Summer School 2018
Nelson NZ
7 to 13 January 2018

Flag varieties
Lecture 1

Arun Ram
University of Melbourne
aram@unimelb.edu.au

$$\begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} =$$



$$\in \quad y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{58}{8}\right) y_2\left(\tfrac{6}{8}\right) y_3\left(\tfrac{11}{5}\right) \mathcal{B}$$

Linear Algebra Theorem 1

(a) $GL_n(\mathbb{C})$ is generated by elementary matrices.

(b) Let

$$\begin{array}{l} G = GL_n(\mathbb{C}) \\ \cup| \\ B = \{\text{upper triangular matrices}\} \end{array}$$

$$W = \{\text{permutation matrices}\}$$

Then

$$G = \bigsqcup_{w \in W} BwB$$

(c) $W = S_n$ is generated by simple transpositions.

(d) The points of G/B in BwB are

$$BwB = \{y_{i_1}(c_1) y_{i_2}(c_2) \cdots y_{i_\ell}(c_\ell) B \mid c_1, c_2, \ldots, c_\ell \in \mathbb{C}\}$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w.

The symmetric group $W = S_n$

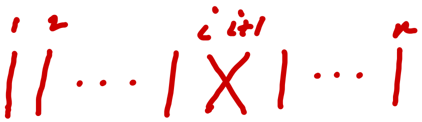
A *permutation* is an $n\times n$ matrix with

(a) exactly one nonzero entry in each row and each column. (Gens A)

(b) The nonzero entries are 1.

The *product* in S_n is given by matrix multiplication. (Rels A)

The *simple transpositions* in S_n are

$s_i =$  $=$

$$\begin{array}{c|c}
 & \begin{matrix} & & & i & i+1 & & & \end{matrix} \\
\begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}
\end{array}$$

(Gens B)

Theorem The symmetric group S_n is presented by generators

$s_1, s_2, \ldots, s_{n-1}$ and relations

$$s_i^2 = 1, \qquad s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}, \qquad s_k s_\ell = s_\ell s_k \qquad \text{(RelsB)}$$

where $i, j, k, \ell \in \{1, \ldots, n-1\}$, $j \neq n-1$, and $k \neq \ell \pm 1$.

Proof: Four steps

(1) Generators B in terms of Generators A

(2) Relations B from Relations A

(3) Generators A in terms of Generators B.

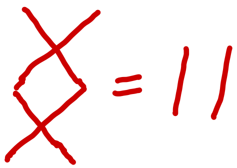
(4) Relations A from Relations B.

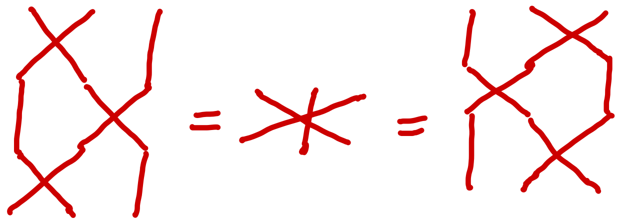
Step (1) Generators B in terms of Generators A

This is given in (GensB)

Step (2) Relations B from Relations A

Do the computations





in terms of matrices.

Step (3) Generators A in terms of Generators B.

Let w be a permutation. Define

$$w^{(1)} = s_1 s_2 \cdots s_{j_1-1} w, \quad \text{where } w(j_1, 1) = 1,$$

$$w^{(2)} = s_2 s_3 \cdots s_{j_2-1} w^{(1)}, \quad \text{where } w^{(1)}(j_2, 2) = 1$$

$$\vdots$$

Then $w^{(n-1)} = 1$, and

$$w = (s_{j_1-1} \cdots s_2 s_1)(s_{j_2-1} \cdots s_3 s_2) \cdots$$

This is a <u>normal form</u> for w

(a favourite reduced word for w).

Step (4) Relations A from Relations B.

Let w_1 and w_2 be permutations written in <u>normal form</u>

To do: Compute $w_1 w_2$ using only Rels B

The crucial step is:

Assuming $k<i<j$ then

$$s_i(s_{j-1}\cdots s_{k+1}s_k) = s_{j-1}\cdots s_{i+2}\, s_i s_{i+1} s_i s_{i-1}\cdots s_{k+1}s_k$$

$$= s_{j-1}\cdots s_{i+2}\, s_{i+1} s_i s_{i+1}\, s_{i-1}\cdots s_{k+1}s_k$$

$$= (s_{j-1}\cdots s_{i+2}\, s_{i+1} s_i s_{i-1}\cdots s_{k+1}s_k)\, s_{i+1}.$$

By this process $w_1 w_2$ can be written in normal form

(one simple transposition at a time)

Using only Rels B.

The permutation computation

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = s_2 s_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= (s_2 s_1)(s_3 s_2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

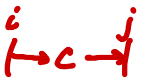
$$= (s_2 s_1)(s_3 s_2) s_3$$

The group $GL_n(\mathbb{C})$

An <u>invertible matrix</u> is an $n \times n$ matrix g such that g^{-1} satisfying $gg^{-1}=1$ and $g^{-1}g=1$ exists. (Gens A)

The <u>product</u> in $GL_n(\mathbb{C})$ is given by matrix multiplication. (Rels A)

The <u>elementary matrices</u> in $GL_n(\mathbb{C})$ are (Gens B)



$$= x_{ij}(c) = \begin{array}{c c} & \begin{array}{ccccc} & & & j & \end{array} \\ i & \begin{pmatrix} 1 & & & c & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array}$$

$$= y_i(c) = \begin{array}{c c} & \begin{array}{cccccc} & & i & i+1 & & \end{array} \\ \begin{array}{c} \\ \\ i \\ i+1 \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix} \end{array}$$

$$= h_i(d) = \begin{array}{c c} & \begin{array}{ccccc} & & i & & \end{array} \\ i & \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & d & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \end{array}$$

Theorem The group $GL_n(\mathbb{C})$ is presented by generators

$$x_{ij}(c), \quad h_i(d), \quad \text{and} \quad y_i(c)$$

and relations

given on the following pages. (RelsB)

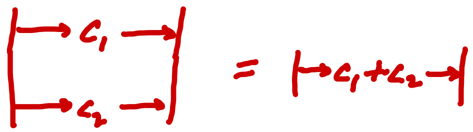
Proof: Four steps

(1) Generators B in terms of Generators A

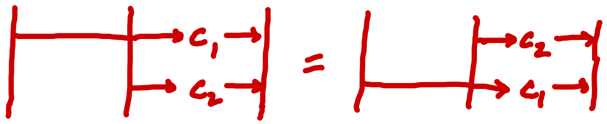
(2) Relations B from Relations A

(3) Generators A in terms of Generators B.

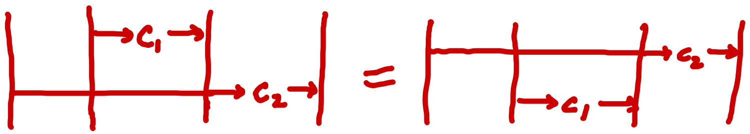
(4) Relations A from Relations B.



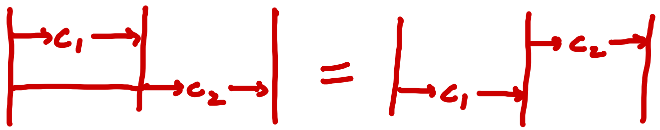
$$x_{ij}(c_1)\, x_{ij}(c_2) = x_{ij}(c_1+c_2)$$



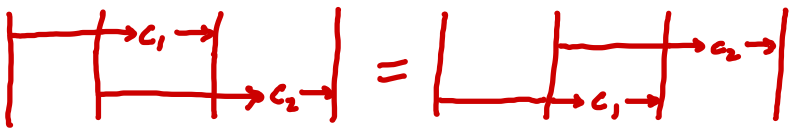
$$x_{ik}(c_1)\, x_{jk}(c_2) = x_{jk}(c_2)\, x_{ik}(c_1)$$



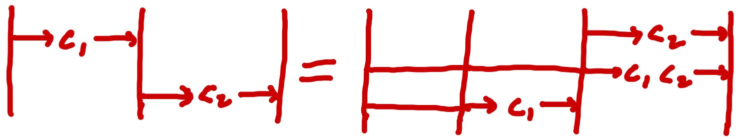
$$x_{jk}(c_1)\, x_{i\ell}(c_2) = x_{i\ell}(c_2)\, x_{jk}(c_1)$$



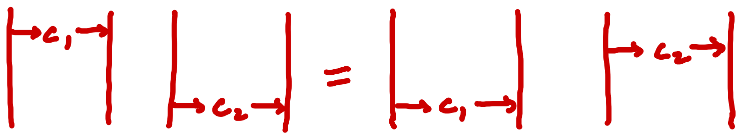
$$x_{jk}(c_1)\, x_{j\ell}(c_2) = x_{j\ell}(c_2)\, x_{ik}(c_1)$$



$$x_{ik}(c_1)\, x_{j\ell}(c_2) = x_{j\ell}(c_2)\, x_{ik}(c_1)$$

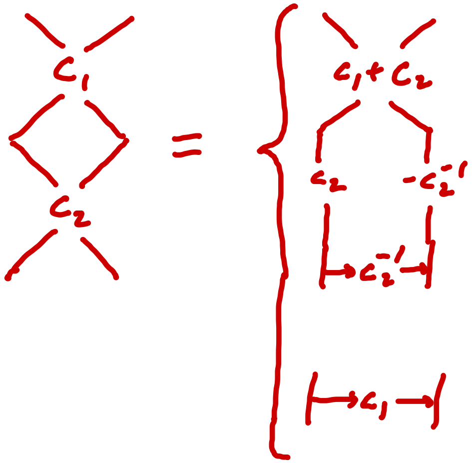


$$x_{ij}(c_1)\, x_{jk}(c_2) = x_{jk}(c_2)\, x_{ik}(c_1 c_2)\, x_{ij}(c_1)$$



$$x_{ij}(c_1)\, x_{k\ell}(c_2) = x_{k\ell}(c_2)\, x_{ij}(c_1)$$

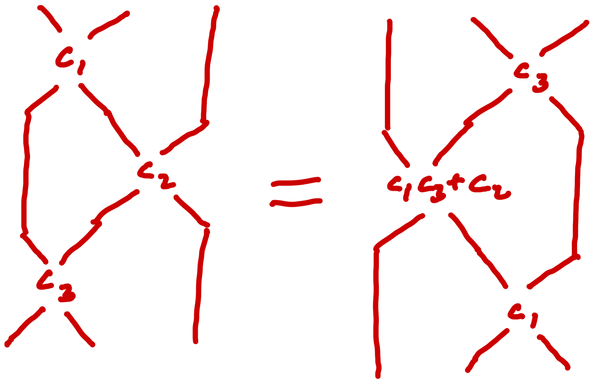
<u>X-relations</u>



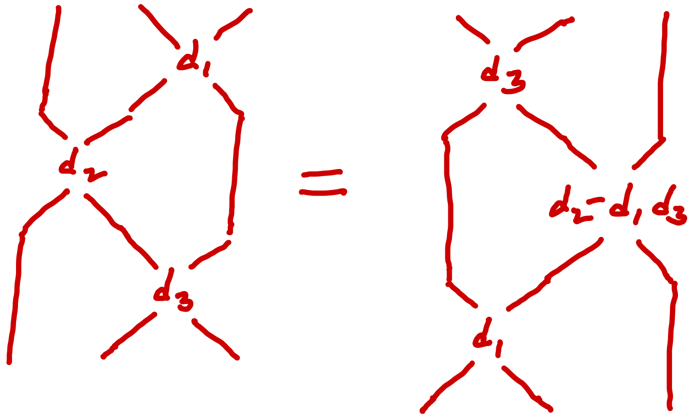
if $c_2 \neq 0$,

if $c_2 = 0$.

$$y_i(c_1)\,y_i(c_2) = \begin{cases} y_i(c_1 + c_2^{-1})\,h_i(c_2)\,h_{i+1}(-c_2^{-1})\,x_{i,i+1}(c_2^{-1}), & \text{if } c_2 \neq 0, \\ x_{i,i+1}(c_1), & \text{if } c_2 = 0. \end{cases}$$

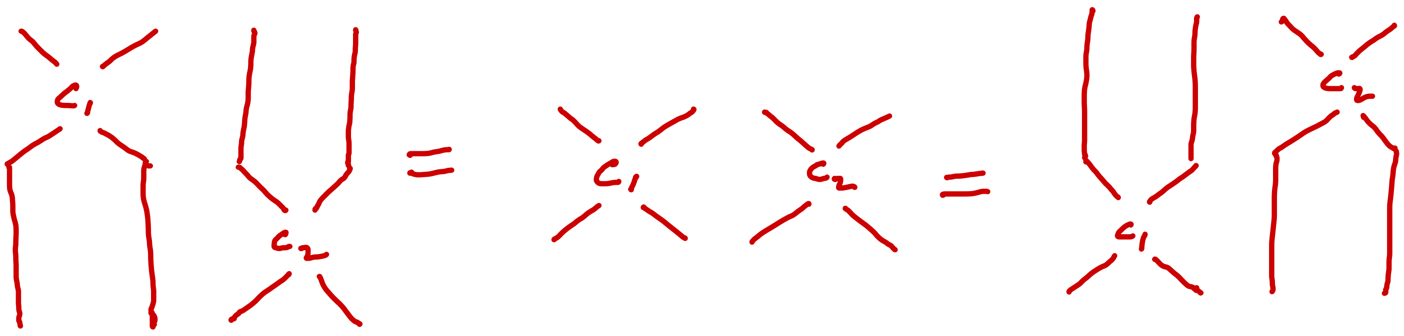


and



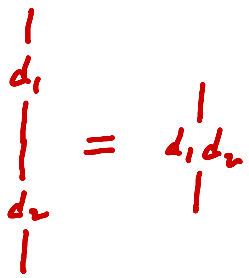
$$y_j(c_1)\,y_{j+1}(c_2)\,y_j(c_3) = y_{j+1}(c_3)\,y_j(c_1c_3+c_2)\,y_{j+1}(c_1) \quad \text{and}$$

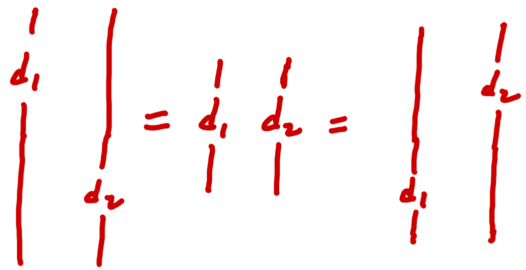
$$y_{j+1}(d_1)\,y_j(d_2)\,y_{j+1}(d_3) = y_j(d_3)\,y_{j+1}(d_2-d_1d_3)\,y_j(d_1)$$



$$y_k(c_1)y_\ell(c_2) = y_\ell(c_2)y_k(c_1)$$

y-relations

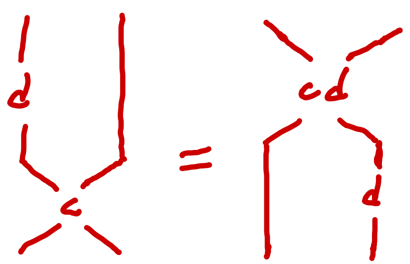




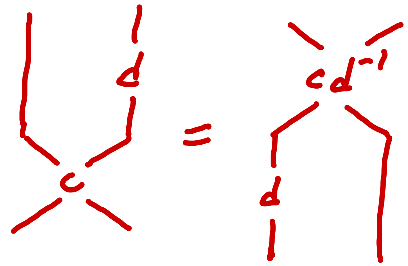
$$h_i(d_1)h_i(d_2) = h_i(d_1 d_2)$$

$$h_i(d_1)h_j(d_2) = h_j(d_2)h_i(d_1)$$

h-relations

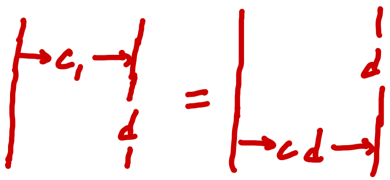


and

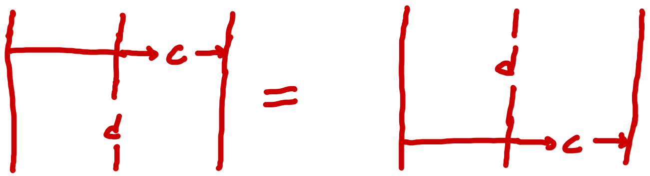


$$h_j(d)\,y_j(c) = y_j(cd)\,h_{j+1}(d) \qquad h_{j+1}(d)\,y_j(c) = y_j(cd^{-1})\,h_j(d)$$

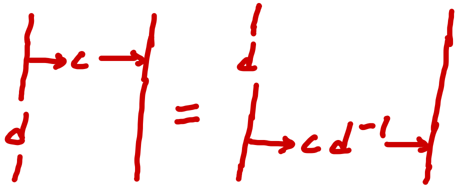
<u>Moving $h_i(d)$ to the right of $y_j(d)$</u>



$$x_{ij}(c)\,h_j(d) = h_j(d)\,x_{ij}(cd)$$

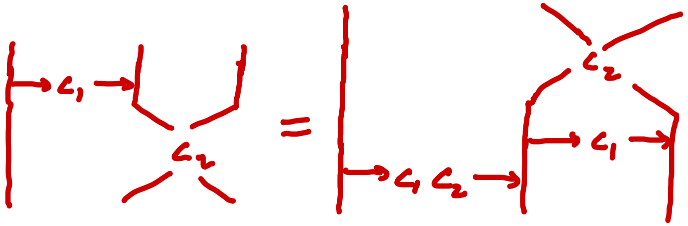


$$x_{ik}(c)\,h_j(d) = h_j(d)\,x_{ik}(c)$$

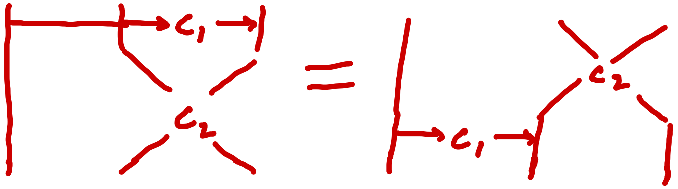


$$x_{ij}(c)\,h_i(d) = h_i(d)\,x_{ij}(cd^{-1})$$

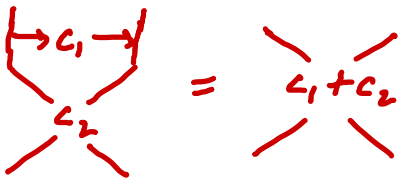
<u>Moving $x_{ij}(c)$ to the right of $h_k(d)$</u>



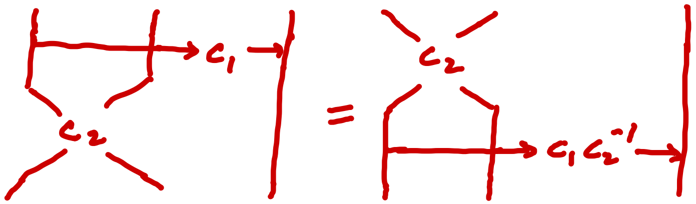
$$x_{ij}(c_1)y_j(c_2) = y_j(c_2)x_{j,j+1}(c_1)x_{ij}(c_1c_2)$$



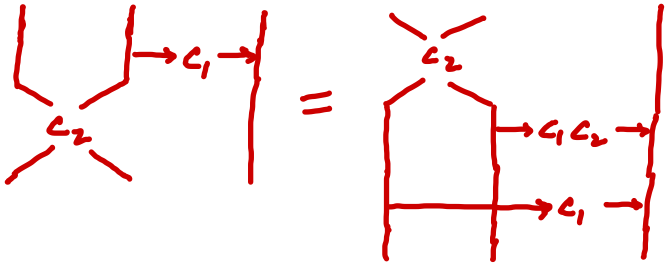
$$x_{i,j+1}(c_1)y_j(c_2) = y_j(c_2)x_{ij}(c_1)$$



$$x_{j,j+1}(c)y_j(c_2) = y_j(c_1+c_2)$$



$$x_{jk}(c_1)y_j(c_2) = y_j(c_2)x_{jk}(c_1c_2^{-1})$$



$$x_{j+1,k}(c_1)y_j(c_2) = y_j(c_2)x_{j+1,k}(c_1c_2)x_{jk}(c_1)$$

<u>Moving $x_{ij}(c_1)$ to the right of $y_k(c_2)$</u>

$$\begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} = y_2\left(\tfrac{1}{8}\right)\begin{pmatrix} 7 & 6 & 2 & 4 \\ 8 & 6 & 3 & 5 \\ 0 & \frac{59}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix} = y_2\left(\tfrac{1}{8}\right)y_1\left(\tfrac{7}{8}\right)\begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{6}{8} & \frac{-5}{8} & \frac{-3}{8} \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right)y_1\left(\tfrac{7}{8}\right)y_3\left(\tfrac{58}{8}\right)\begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{6}{8} & \frac{-5}{8} & \frac{-3}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \frac{-5}{8} & \frac{-49}{8} \end{pmatrix} = y_2\left(\tfrac{1}{8}\right)y_1\left(\tfrac{7}{8}\right)y_3\left(\tfrac{58}{8}\right)y_2\left(\tfrac{6}{8}\right)\begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \frac{-11}{8} & \frac{-15}{8} \\ 0 & 0 & \frac{-5}{8} & \frac{-49}{8} \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right)y_1\left(\tfrac{7}{8}\right)y_3\left(\tfrac{58}{8}\right)y_2\left(\tfrac{6}{8}\right)y_3\left(\tfrac{11}{5}\right)\begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \frac{-5}{8} & \frac{-49}{8} \\ 0 & 0 & 0 & \frac{464}{40} \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right)y_1\left(\tfrac{7}{8}\right)y_3\left(\tfrac{58}{8}\right)y_2\left(\tfrac{6}{8}\right)y_3\left(\tfrac{11}{5}\right)h_1(8)\,h_2(1)\,h_3\left(\tfrac{-5}{8}\right)h_4\left(\tfrac{464}{40}\right)\begin{pmatrix} 1 & \frac{6}{8} & \frac{3}{8} & \frac{5}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{49}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{6}{8} & \frac{3}{8} & \frac{5}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{49}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix} = x_{34}\left(\tfrac{49}{5}\right) x_{24}(2)\, x_{14}\left(\tfrac{5}{8}\right) \begin{pmatrix} 1 & \frac{6}{8} & \frac{3}{8} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

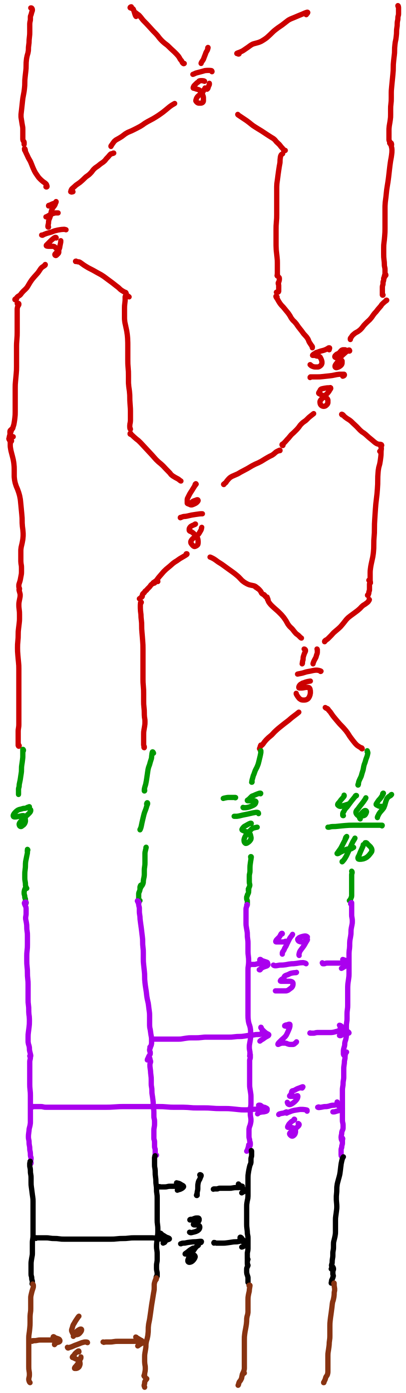
$$= x_{34}\left(\tfrac{49}{5}\right) x_{24}(2)\, x_{14}\left(\tfrac{5}{8}\right) x_{23}(1)\, x_{13}\left(\tfrac{3}{8}\right) \begin{pmatrix} 1 & \frac{6}{8} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= x_{34}\left(\tfrac{49}{5}\right) x_{24}(2)\, x_{14}\left(\tfrac{5}{8}\right) x_{23}(1)\, x_{13}\left(\tfrac{3}{8}\right) x_{12}\left(\tfrac{6}{8}\right)$$

$$\begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} = y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{58}{8}\right) y_2\left(\tfrac{6}{8}\right) y_3\left(\tfrac{11}{5}\right) h_1(8)\, h_2(1)\, h_3\left(\tfrac{-5}{8}\right) h_4\left(\tfrac{464}{40}\right)$$

$$\cdot\, x_{34}\left(\tfrac{49}{5}\right) x_{24}(2)\, x_{14}\left(\tfrac{5}{8}\right) x_{23}(1)\, x_{13}\left(\tfrac{3}{8}\right) x_{12}\left(\tfrac{6}{8}\right)$$

$$\begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} =$$



$$\in \; y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{58}{8}\right) y_2\left(\tfrac{6}{8}\right) y_3\left(\tfrac{11}{5}\right) \mathcal{B}$$