<u>Schubert Calculus for semi-infinite flag varieties</u> 07.10.2018

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<u>Schubert Calculus</u> is the study of $K_B(G/B)$,

the cohomology of the <u>flag variety</u> G/B,

in the basis of <u>Schubert classes</u> $\{[\overline{BwB}] \mid w \in W\}$.

G = Kac-Moody group
$\cup$|
B = Borel subgroup W = Weyl group

<u>Example</u> $G = GL_n(\mathbb{C})$, $W = S_n$ and

$$B = \left\{ \begin{pmatrix} a_1 & & u_{ij} \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \;\middle|\; \begin{matrix} u_{ij} \in \mathbb{C} \\ a_i \in \mathbb{C}^\times \end{matrix} \right\}$$

Row reduction provides a <u>cell decomposition</u> of G/B

$$G = \bigsqcup_{w \in W} BwB \quad \text{with} \quad BwB \cong \mathbb{C}^{\ell(w)}$$

(as a subset of G/B) defining the <u>length function</u> on W.

$$\overline{BwB} = \bigsqcup_{v \le w} BvB \qquad \left(\begin{matrix}\text{closure order} \\ \text{on } W\end{matrix}\right)$$

Then $K_B(G/B)$ is a $K_B(pt)$-algebra

with basis $\{[\overline{BwB}] \mid w \in W\}$.

Affine flag varieties

G is an affine Kac-Moody group with

Lie algebra $\mathfrak{g} = (\mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon,\epsilon^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d$

where $\mathring{\mathfrak{g}}$ is a fin. dim'l simple Lie algebra.

Three! flag varieties

G/I^+	G/I^0	G/I^-
positive level	level zero	negative level
= thin	= semiinfinite	= thick

Example $\mathring{\mathfrak{g}} = \mathfrak{sl}_n$, $G = SL_n(\mathbb{C}[\epsilon,\epsilon^{-1}])$ or $SL_n(\mathbb{C}((\epsilon)))$

$$I^+ = \left\{ \begin{pmatrix} a_1 & & u_{ij} \\ & \ddots & \\ v_{ij} & & a_n \end{pmatrix} \;\middle|\; \begin{matrix} u_{ij} \in \mathbb{C}[[\epsilon]] \\ a_i \in \mathbb{C}[[\epsilon]]^\times \\ v_{ij} \in \epsilon\mathbb{C}[[\epsilon]]^* \end{matrix} \right\}$$

$$I^0 = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ v_{ij} & & a_n \end{pmatrix} \;\middle|\; \begin{matrix} a_i \in \mathbb{C}[[\epsilon]]^\times \\ v_{ij} \in \mathbb{C}((\epsilon)) \end{matrix} \right\}$$

$$I^- = \left\{ \begin{pmatrix} a_1 & & u_{ij} \\ & \ddots & \\ v_{ij} & & a_n \end{pmatrix} \;\middle|\; \begin{matrix} u_{ij} \in \epsilon^{-1}\mathbb{C}[[\epsilon^{-1}]] \\ a_i \in \mathbb{C}[[\epsilon^{-1}]]^\times \\ v_{ij} \in \mathbb{C}[[\epsilon^{-1}]] \end{matrix} \right\}$$

Case 1:

$\mathbb{F} = \mathbb{C}((\epsilon))$

$\mathfrak{o} = \mathbb{C}[[\epsilon]]$

$\mathfrak{o}^\times = \mathbb{C}[[\epsilon]]^\times$

$\mathfrak{o}^- = \mathbb{C}[\epsilon^{-1}]$

$(\mathfrak{o}^-)^\times = \mathbb{C}[\epsilon^{-1}]^\times = \mathbb{C}^\times$

Case 2

$\mathbb{F} = \mathbb{C}[\epsilon,\epsilon^{-1}]$

$\mathfrak{o} = \mathbb{C}[\epsilon]$

$\mathfrak{o}^\times = \mathbb{C}[\epsilon]^\times = \mathbb{C}^\times$

$\mathfrak{o}^- = \mathbb{C}[\epsilon^{-1}]$

$(\mathfrak{o}^-)^\times = \mathbb{C}[\epsilon^{-1}]^\times = \mathbb{C}^\times$.

Cell decompositions

Let W be the affine Weyl group.

$$G = \bigsqcup_{x \in W} I^+ x I^+ \qquad G = \bigsqcup_{y \in W} I^+ y I^0 \qquad G = \bigsqcup_{z \in W} I^+ z I^-$$

Remark $I^+ x I^+ \cong \mathbb{C}^{\ell(x)}$ in G/I^+, but

$I^+ y I^0 \cong \mathbb{C}^\infty$ in G/I^0 and $I^+ z I^- \cong \mathbb{C}^\infty$ in G/I^-.

Closure orders

$$\overline{I^+ x I^+} = \bigsqcup_{w \leq_+ x} I^+ w I^+$$

$$\overline{I^+ y I^0} = \bigsqcup_{w \leq_0 y} I^+ w I^0$$

$$\overline{I^+ z I^-} = \bigsqcup_{w \leq_- y} I^+ w I^-$$

As a $K_{I^+}(pt)$ module

$K_{I^+}(G/I^+)$ has basis $\{[\overline{I^+ x I^+}] \mid x \in W\}$

$K_{I^+}(G/I^0)$ has basis $\{[\overline{I^+ y I^0}] \mid y \in W\}$

$K_{I^+}(G/I^-)$ has basis $\{[\overline{I^+ z I^-}] \mid z \in W\}$.

Moment graph presentation of $K_B(G/B)$

Let $G = GL_n(\mathbb{C})$ with $W = S_n$ and

$$T = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \;\middle|\; a_i \in \mathbb{C}^\times \right\}$$

Then

$$K_B(G/B) \cong K_T(G/B) \xrightarrow{\Phi} K_T((G/B)^T) = \bigoplus_{w \in W} K_T(pt).$$

Let

$$S = K_T(pt) = \mathbb{Z}[[y_\lambda \mid \lambda \in \mathbb{Z}^n]] = \mathbb{Z}[[y_{\varepsilon_1}, \dots, y_{\varepsilon_n}]]$$

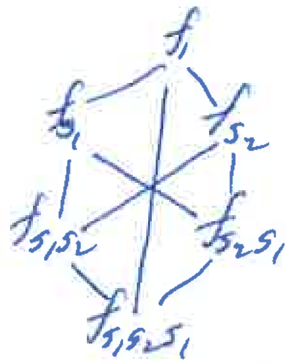
with $y_{\lambda+\mu} = y_\lambda + y_\mu - y_\lambda y_\mu$ and $\varepsilon_i = (0, \dots, 0, \overset{i\text{th}}{1}, 0, \dots 0)$

Let $s_{ij} \in W$ be the transposition switching i and j.

$$K_B(G/B) \cong \operatorname{im}\Phi = \left\{ f = (f_w)_{w \in W} \;\middle|\; \begin{array}{l} f_w \in S \text{ and} \\ f_w - f_{ws_{ij}} \in y_{\varepsilon_i - \varepsilon_j} S \end{array} \right\}$$

Example $n = 3$

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1,\; s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$$

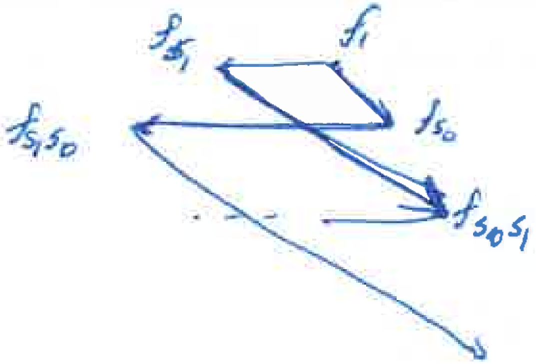


with

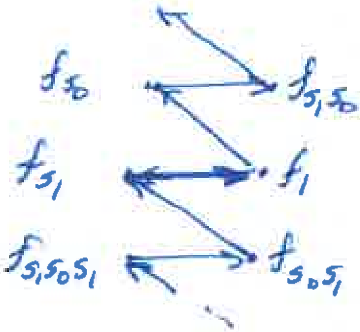
$$f_1 - f_{s_1} \in y_{\varepsilon_1 - \varepsilon_2} S$$
$$f_1 - f_{s_2} \in y_{\varepsilon_2 - \varepsilon_3} S$$
$$f_1 - f_{s_1 s_2 s_1} \in y_{\varepsilon_1 - \varepsilon_3} S$$
$$\vdots$$

etc

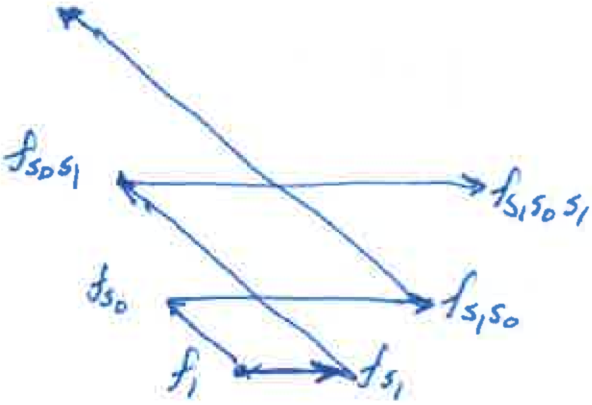
For $G = \widehat{SL_2}(\mathbb{C}((\epsilon)))$, $W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1 \rangle$



positive level
$K_{I^+}(G/I^+)$



level zero
$K_{I^+}(G/I^0)$



negative level
$K_{I^+}(G/I^-)$.

The affine Hecke algebra action

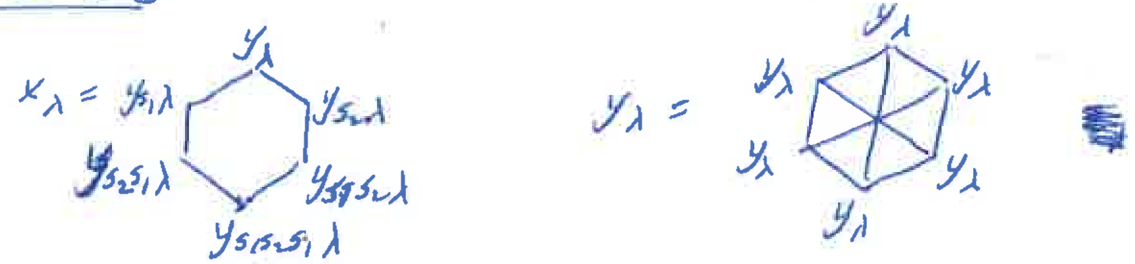
Define a ~~(hopefully surjective (usually it is))~~ surjective homomorphism

$$S \otimes S \longrightarrow K_{I^+}(G/I^0) \quad \text{by}$$

$$y_\lambda \otimes 1 \longmapsto x_\lambda = (y_{w^{-1}\lambda})_{w \in W}$$

$$1 \otimes y_\lambda \longmapsto y_\lambda = (y_\lambda)_{w \in W}.$$

For $G = GL_3$



A simple reflection is $s \in W$ such that

$$I^+ s I^+ \cong \mathbb{C}. \quad \text{Then} \quad P_s = I^+ \sqcup I^+ s I^+$$

is a subgroup $P_s \ni I$, giving "change of group" maps z_s and z^s

$$T_s : K_{I^+}(G/I^0) \xrightarrow{z_s} K_{P_s}(G/I^0) \xrightarrow{z^s} K_{I^s}(G/I^0)$$

Let $z_\lambda : K_{I^+}(G/I^0) \to K_{I^+}(G/I^0)$ be multiplication by z_λ.

$t_s : K_{I^+}(G/I^+) \longrightarrow K_{I^+}(G/I^0)$ given by $t_s(f_w) = (f_{sw})_{w \in W}$.

Proposition (with, perhaps, some slight corrections).

(a) T_s acts by $(t_s + 1)\frac{1}{z_\lambda}$

(b) These operators provide an action of

$$K_{I^+}(G/I^+) \quad \text{on} \quad K_{I^+}(G/I^0)$$

Representation theory

In $K_{I^+}(G/I^0)$ the line bundle $\mathcal{L}_\lambda = G \times_{I^0} \mathbb{C}_\lambda$ corresponding to the 1 dim'l I^+-module $\mathbb{C}_\lambda$ has

$[\mathcal{L}_\lambda] = 1 - x_\lambda$. Let $e^\lambda = 1 - y_\lambda$

(Note: $e^\lambda e^\mu = (1-y_\lambda)(1-y_\mu) = (1 - y_\lambda - y_\mu + y_\lambda y_\mu) = 1 - y_{\lambda+\mu} = e^{\lambda+\mu}$).

Let

$\pi : G/I^0 \longrightarrow pt$ and $\pi_! : K_{I^+}(G/I^0) \longrightarrow K_{I^+}(pt)$.

Then the I^+-module

$$\pi_!([\mathcal{L}_\lambda][\overline{I^+ z I^0}]) = H^0(\overline{I^+ z I^0}, \mathcal{L}_\lambda) = L(\lambda + 0\Lambda_0)_{\geq_0 z}$$

is a Demazure submodule of the G-module

$$H^0(G/I^0, \mathcal{L}_\lambda) = L(\lambda + 0\Lambda_0) \quad \left(\begin{matrix}\text{extremal weight}\\ \text{module}\end{matrix}\right)$$

<u>Theorem</u> (Kato-Naito-Sagaki) In $K_{I^+}(G/I^0)$

(a) $$[\mathcal{L}_\lambda \otimes \mathcal{O}_{\overline{I^+ z I^0}}] = \sum_{w \in W} c^w_{\lambda, z} [\overline{I^+ w I^0}]$$

$$\Longleftrightarrow H^0(\overline{I^+ z I^0}, \mathcal{L}_\lambda \otimes \mathcal{L}_\mu) = \sum_{w \in W} c^w_{\lambda, z} H^0(\overline{I^+ w I^0}, \mathcal{L}_\mu)$$

(b) $$[\mathcal{L}_\lambda][\overline{I^+ z I^0}] = \sum_{p \in \mathbb{B}^{\infty/2}_{\geq_0 z}(\lambda)} e^{wt(p)} [\overline{I^+ z(p) I^0}]$$

$\mathbb{B}^{\infty/2}(\lambda)$ is the crystal of $L(\lambda + 0\Lambda_0)$

$\mathbb{B}^{\infty/2}_{\geq_0 z}(\lambda)$ is the crystal of $L(\lambda + 0\Lambda_0)_{\geq_0 z}$.