Geometric and Categorical Representation Theory, Creswick, MATRIX ①

TBTL **Two Boundary Temperley-Lieb** 18.12.2018

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and the **exotic nilcone**.

Parameters: $t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_k^{\frac{1}{2}}$.

$$q_1 = t^{v_1} = t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}}, \quad q_2 = +t^{v_2} = t_0^{\frac{1}{2}} t_k^{\frac{1}{2}}, \quad q = t^{\frac{1}{2}}$$

Dynkin diagram type $C_n^{(1)}$

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1, \quad T_0^2 = (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})T_0 + 1$$
$$T_k^2 = (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})T_k + 1$$

} Hecke algebra of type $C_n^{(1)}$ with unequal parameters

Put

$e_i = T_i - t^{\frac{1}{2}}$, $e_0 = T_0 - t_0^{\frac{1}{2}}$ and $e_k = T_k - t_k^{\frac{1}{2}}$

$e_i e_{i\pm 1} e_i = e_i$; $e_1 e_0 e_1 = (\text{const}) e_1$, $e_{k-1} e_k e_{k-1} = (\text{const}) e_1$

} TL of type $C_n^{(1)}$ = TBTL.

Basis of TBTL

Noncrossing diagrams with

$(-1)^{\#\text{left bdy}} = 1$, $(-1)^{\#\text{right bdy}} = 1$, $(-1)^{\#\text{left bdy to right bdy}} = 1$.

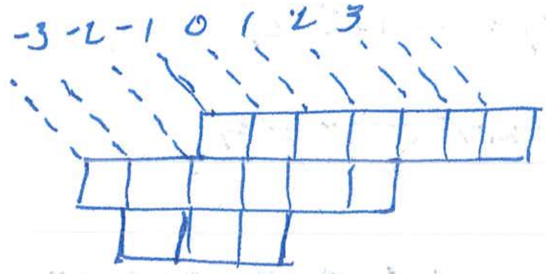
Representations

$$\left\{ \begin{matrix} \text{Irred. reps of} \\ H \text{ of type } A^{(1)}_{k-1} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{multisegments } \lambda \\ \text{with } k \text{ boxes} \end{matrix} \right\}$$

$$\left\{ \begin{matrix} \text{Irred reps of} \\ \mathcal{T} \text{ of type } A^{(1)}_{k-1} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{multisegments } \lambda \\ \text{with } k \text{ boxes and } \le 2 \text{ rows} \end{matrix} \right\}$$

$$\left\{ \begin{matrix} \text{Irred. reps of} \\ H \text{ of type } C^{(1)}_{k} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{marked multisegments } \lambda \\ \text{with } 2k \text{ boxes} \end{matrix} \right\}$$

$$\left\{ \begin{matrix} \text{Irred reps of} \\ \mathcal{T} \text{ of type } C^{(1)}_{k} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{marked multisegments} \\ \text{with } 2k \text{ boxes and } \le 2 \text{ rows} \end{matrix} \right\}$$



Order the boxes. Let $s_i = q^{\text{diag no of box}_i}$.

$$s = (s_1, \ldots, s_k) = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_k \end{pmatrix} \text{ is a semisimple element in } GL_k(\mathbb{C})$$

$n = \sum' E_{ij}$ is a nilpotent element in ~~GL_k~~ $\mathfrak{gl}_k$

and $s n s^{-1} = q n$

The exotic nilcone $\mathcal{N}^{ex}$.

$G = Sp_{2k}(\mathbb{C})$
$\cup\!|$
B

G/B is the flag variety.

Adjoint rep

$\mathfrak{g} = L(2\omega_1)$

$\mathfrak{n}_+ = \bigoplus_{\lambda>0} \mathfrak{g}_\lambda$

Exotic adjoint rep.

$\mathfrak{g}^{ex} = L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_2)$

$\mathfrak{n}_+^{ex} = \bigoplus_{\lambda>0} \mathfrak{g}_\lambda$

where $\lambda > 0$ means $\lambda \in \mathbb{R}_{>0}\text{-span}\{\alpha_1, \dots, \alpha_k\}$.

The Springer resolution

$$\tilde{\mathcal{N}}^{ex} = G \times_B \mathfrak{n}_+^{ex}$$

$\pi: \tilde{\mathcal{N}}^{ex} \to G/B$, $\quad \mu: \tilde{\mathcal{N}}^{ex} \to \mathcal{N}^{ex} = G \cdot \mathfrak{n}_+^{ex}$

$[g, n] \mapsto gB$, $\quad [g, n] \mapsto gn$

The Steinberg variety

$$Z^{ex} = \tilde{\mathcal{N}}^{ex} \times_{\mathcal{N}^{ex}} \tilde{\mathcal{N}}^{ex} = \{(x_1, x_2) \in \tilde{\mathcal{N}}^{ex} \times \tilde{\mathcal{N}}^{ex} \mid \mu(x_1) = \mu(x_2)\}$$

Theorem (Kato)

$$K_{G \times (\mathbb{C}^\times)^3}(Z^{ex}) \cong H \text{ as a } K_{G \times (\mathbb{C}^\times)^3}(pt)\text{-module}$$

Note: $K_{G \times (\mathbb{C}^\times)^3}(pt) \cong Z(H)$.

Generalised Springer fibers $\mathcal{B}_{a,x}$

$(\mathbb{C}^\times)^3 = \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times$ acts on $L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_2) = \mathfrak{g}^{ex}$:

$$\vec{q}^{-1} n = (q_1^{-1}, q_2^{-1}, q^{-1})(n^{(1)} + n^{(2)} + n^{(3)}) = q_1^{-1} n^{(1)} + q_2^{-1} n^{(2)} + q^{-1} n^{(3)}.$$

$G \times (\mathbb{C}^\times)^3$ action:

on G/B: $(s, \vec{q})\, gB = sgB$

on $\mathcal{N}^{ex}$: $(s, \vec{q})\, n = \vec{q}^{-1} s n$

on $\tilde{\mathcal{N}}^{ex}$: $(s, \vec{q})[g, n] = [sg, \vec{q}^{-1} n]$.

<u>Fixed points</u>: Let $a = (s, q_1, q_2, q)$ with s semisimple.

$$\begin{array}{ccc} & (\tilde{\mathcal{N}}^{ex})^a & \\ \swarrow^{\pi^a} & & \searrow^{\mu^a} \\ (G/B)^a & & (\mathcal{N}^{ex})^a \end{array}$$

where $X^a = \{x \in X \mid ax = x\}$.

Let $x \in (\mathcal{N}^{ex})^a$. The <u>generalised Springer fibre</u> is

$$\mathcal{B}_{a,x} = \pi^a((\mu^a)^{-1}(x)).$$

<u>Theorem</u> (Kato)

(a) $K(\mathcal{B}_{a,x})$ is an H-module with simple head $L_{a,x}$

(b)
$$\left\{ \begin{array}{c}\text{irred. } H\text{-modules} \\ L_{a,x}\end{array} \right\} \longleftrightarrow \left\{ (a,x) \,\middle|\, \begin{array}{l} a \in G^{ss} \times (\mathbb{C}^\times)^3 \\ x \in (\mathcal{N}^{ex})^a \end{array} \right\} \Big/ \sim$$

where $(gag^{-1}, gx) \sim (a, x)$ for $g \in G$