A point of view on CFT — Affine Lie algebras

Subfactors in Sydney Conference 4-8 February 2014 ①

$\mathring{\mathfrak{g}}$ finite dim'l simple Lie algebra over $\mathbb{C}$.

$\cup|$

$\mathfrak{a}$ Cartan subalgebra ($\mathfrak{a}^*$ indexes representations of $\mathfrak{a}$ and $\mathring{\mathfrak{g}}$)

The affine Lie algebra is

$$\mathfrak{g} = \mathbb{C}d \oplus (\mathring{\mathfrak{g}} \otimes \mathbb{C}[z,z^{-1}]) \oplus \mathbb{C}K$$

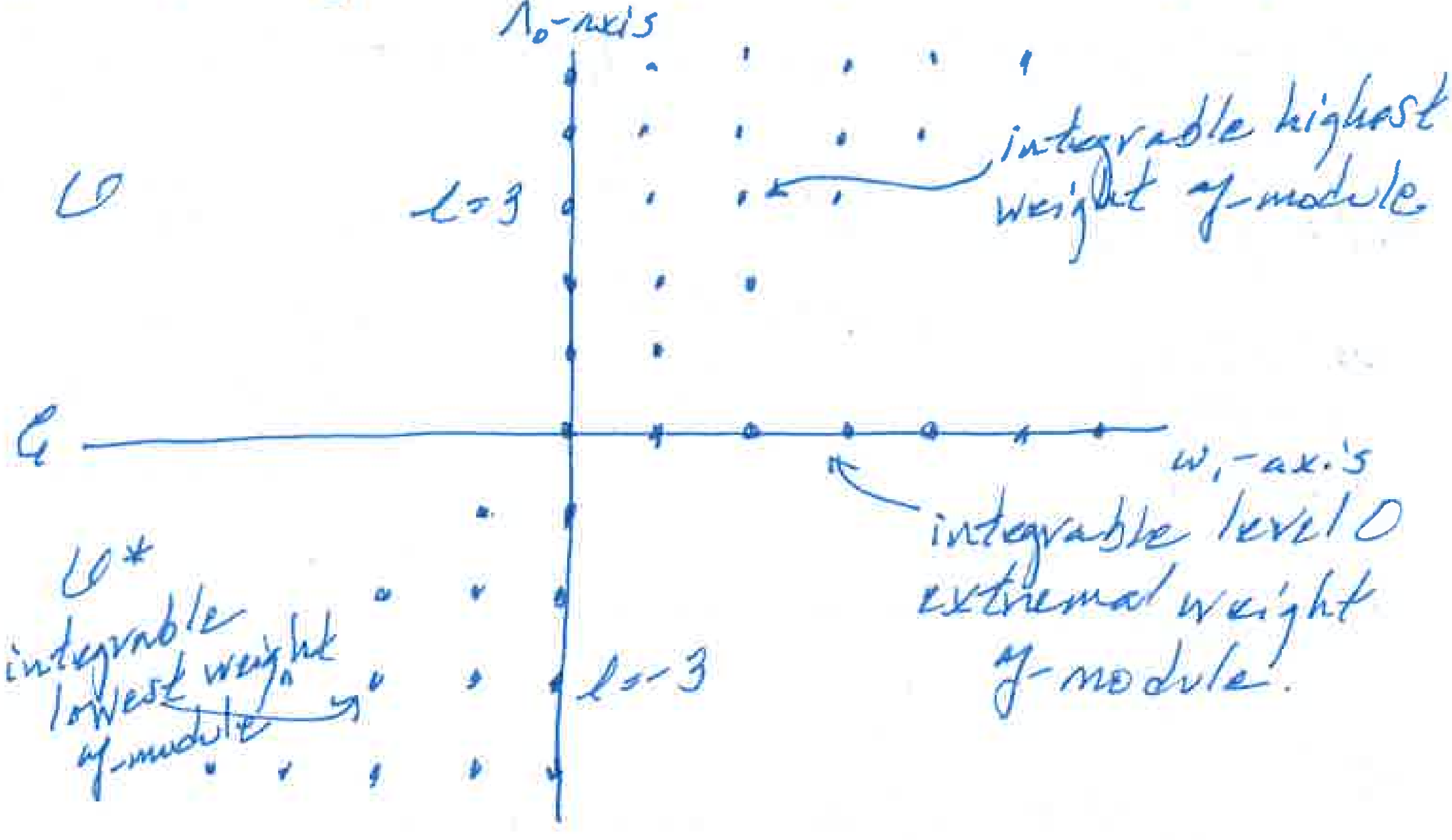
$\cup|$

$$\mathfrak{h} = \mathbb{C}d \oplus \mathfrak{a} \oplus \mathbb{C}K$$

$$\mathfrak{h}^* = \mathbb{C}\delta \oplus \mathfrak{a}^* \oplus \mathbb{C}\Lambda_0 = \mathrm{span}\{\delta, \overbrace{\omega_1, \dots, \omega_n}^{\text{fund wts for } \mathring{\mathfrak{g}}}, \Lambda_0\}$$

Integrable representations

Example $\mathring{\mathfrak{g}} = \mathfrak{sl}_2$, $\mathfrak{h}^* = \mathrm{span}\{\delta, \omega_1, \Lambda_0\}$

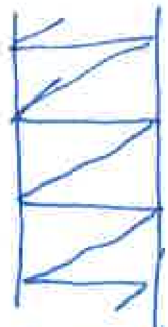


$L(\lambda)$ have <u>characters</u>:

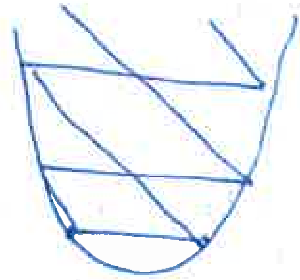
Weyl-Kac character for $\mathcal{O}$	Macdonald polynomials for $\mathcal{C}$	Weyl-Kac character for $\mathcal{O}^*$

$L(\lambda)$ have <u>crystals</u>: (in the sense of quantum groups)
(combinatorics)

highest weight



level 0
$\mathfrak{g}$-module



lowest weight.

$L(\lambda)$ have <u>geometry</u> (Borel-Weil-Bott theorem)

$G/I^+ = \bigsqcup_w I^+ w I^+$	$G/I^0 = \bigsqcup_w I^+ w I^0$	$G/I^- = \bigsqcup_w I^+ w I^-$
affine flag variety pos. level	semi-infinite flag variety level 0	thick affine flag variety neg. level.

If $L(\mu)$ is level m and
$L(\lambda)$ is level n
then $L(\mu) \otimes L(\lambda)$ is level $m+n$.

$\mathcal{C}$ is a <u>tensor category</u>.

<u>Translation functors</u> (for $\mathring{\mathfrak{g}}$ 1970s - ....) ③

$$\mathcal{O} = \bigoplus_{\lambda} \mathcal{O}^{[\lambda]} \quad \text{(block decomposition)}.$$

category of highest weight $\mathring{\mathfrak{g}}$-module
(e.g. Verma modules)

The <u>translation functor</u>

$$T_{\mu}^{\lambda} : \mathcal{O}^{[\mu]} \longrightarrow \mathcal{O}^{[\lambda]}$$

is tensoring by $L(\lambda-\mu)$ a <u>fin</u> <u>dim'l</u> $\mathring{\mathfrak{g}}$-module

$$\underset{\text{Verma}}{M(\mu)} \otimes \underset{\text{fin.dim'l}}{L(\lambda-\mu)} =$$

The "wall crossing" functors

$$T_{\lambda}^{s_p \circ \lambda} : \mathcal{O}^{[\lambda]} \longrightarrow \mathcal{O}^{[s_p \circ \lambda]} = \mathcal{O}^{[\lambda]}$$

give an action of the <u>Hecke</u> <u>algebra</u>
on the Grothendieck group of $\mathcal{O}^{[\lambda]}$.

- This is the source of Kazhdan-Lusztig theory
- $M(\mu) \otimes L(\lambda-\mu)$ is studied by

Shapovalov determinants, Jantzen determinants
Kac-Kazhdan determinants.

Vertex operators

Data: $L(\mu)$, $L(\lambda)$ are highest wt. $\mathfrak{g}$-modules.

W a finite dim'l $\mathfrak{g}'$-module

$W^{aff} = W \otimes \mathbb{C}[z, z^{-1}]$, a $\mathfrak{g}$ module

($\mathfrak{g}'$ is $\mathfrak{g}$ without d).

A <u>vertex operator</u> is a $\mathfrak{g}$-module morphism

$$\Phi : L(\mu) \longrightarrow L(\lambda) \otimes W^{aff}$$

This gives an action of

$\mathcal{C}$ level 0 integrable $\mathfrak{g}$-modules

on $\mathcal{O}$ highest weight $\mathfrak{g}$-modules

$$\mathcal{O} \otimes \mathcal{C} \longrightarrow \mathcal{O}$$

and

$$\mathcal{O} \otimes \underbrace{\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \otimes \cdots \otimes \mathcal{C}}_{k\text{-factors}} \otimes \mathcal{O}^* \longrightarrow \mathcal{O}$$

is a "<u>k-point function</u>".