Combinatorics at level 0

Representation Theory Seminar, Univ. Melbourne
19.02.2019 A. Ram

The affine Lie algebra

$$\mathfrak{g} = (\mathring{\mathfrak{g}} \otimes \mathbb{C}[\epsilon,\epsilon^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

$\mathring{\mathfrak{g}}$ is a fin. dim'l simple Lie algebra over $\mathbb{C}$

$[K, x\epsilon^k] = 0$ and $[K,d] = 0$ for $x \in \mathring{\mathfrak{g}}$

$[d, x\epsilon^l] = l\, x\epsilon^l$

$[x\epsilon^l, y\epsilon^m] = [x,y]\epsilon^{l+m}$ if $m \neq -l$

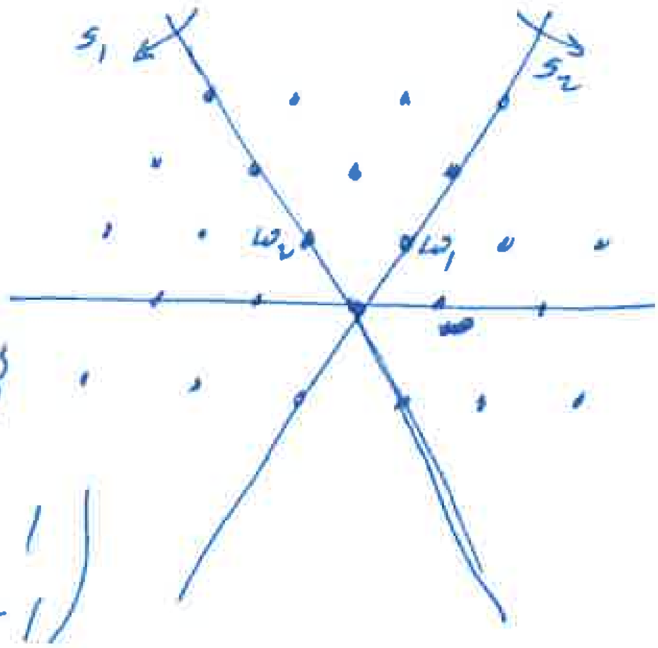
$[x\epsilon^l, y\epsilon^{-l}] = [x,y] + \langle x,y\rangle K.$

Representations of $\mathring{\mathfrak{g}}$ Indexed by dominant integral weights

$\mathring{\mathfrak{g}} = \mathfrak{sl}_2$

$W_0 = \{1, s_1\}$ with $s_1 = (-1)$.

$\mathring{\mathfrak{g}} = \mathfrak{sl}_3$



$$W_0 = \left\langle s_1, s_2 \;\middle|\; \begin{matrix} s_1s_2s_1 = s_2s_1s_2 \\ s_1^2 = 1,\ s_2^2 = 1 \end{matrix} \right\rangle$$

$$= \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$$

$$s_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

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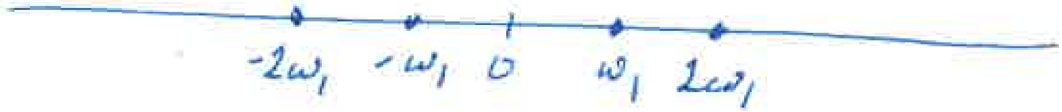
The adjoint representation $L = \mathring{\mathfrak{g}}$ ($\mathring{\mathfrak{g}}$ acts on itself)

$\mathring{\mathfrak{g}} = sl_2 = \mathrm{span}\{e, f, h\}$ with $[e,f] = h$
$[h,f] = -2f \quad [h,e] = 2e$.

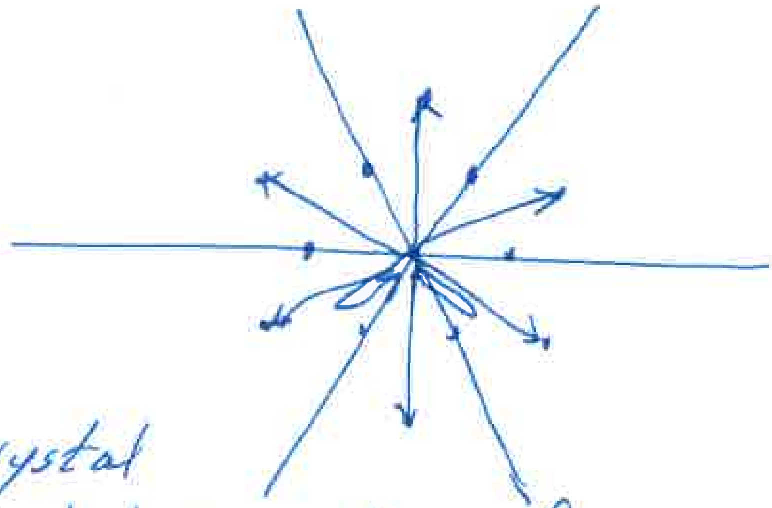


Crystal $\mathcal{B}(2\omega_1) = \{ \longrightarrow, \circlearrowright, \longleftarrow \}$

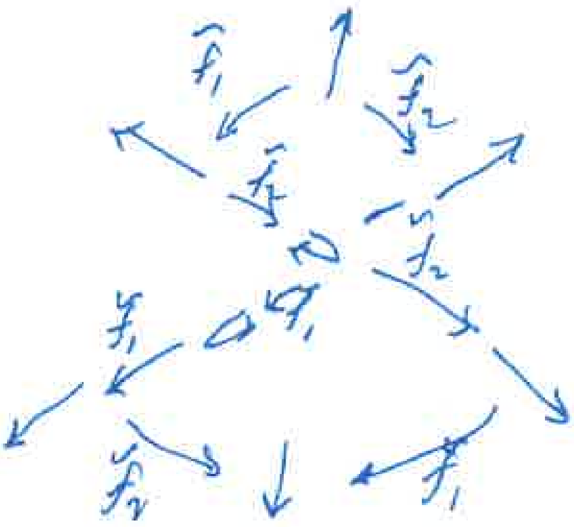
$\tilde{f}(\longrightarrow) = \circlearrowright, \quad \tilde{f}(\circlearrowright) = \longleftarrow, \quad \tilde{f}(\longleftarrow) = 0$

$\tilde{e}(\longrightarrow) = 0, \quad \tilde{e}(\circlearrowright) = \longrightarrow, \quad \tilde{e}(\longleftarrow) = \circlearrowright$.

$\mathring{\mathfrak{g}} = sl_3$ has $\dim(L) = 8$, $\quad L = L(\omega_1 + \omega_2)$



The crystal
$B(\omega_1 + \omega_2)$ has action of $\tilde{e}_1, \tilde{e}_2, \tilde{f}_1, \tilde{f}_2$



Highest weight integrable Representations of $\mathfrak{g}$ Indexed by dominant integral weights

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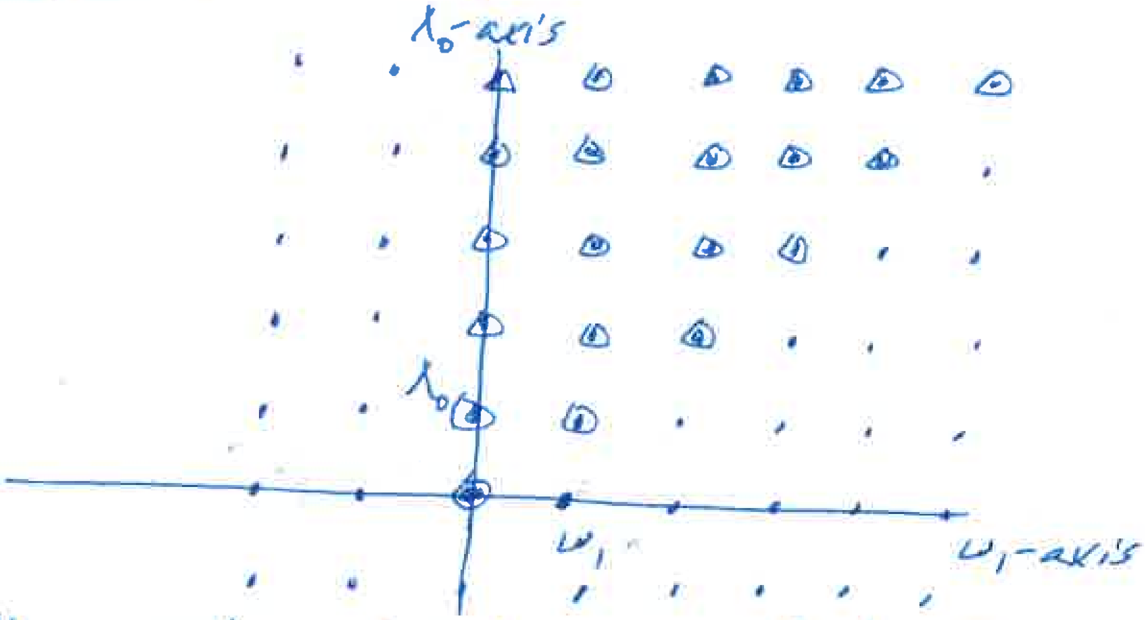
Example $\mathfrak{g}$ when $\mathring{\mathfrak{g}} = sl_2$



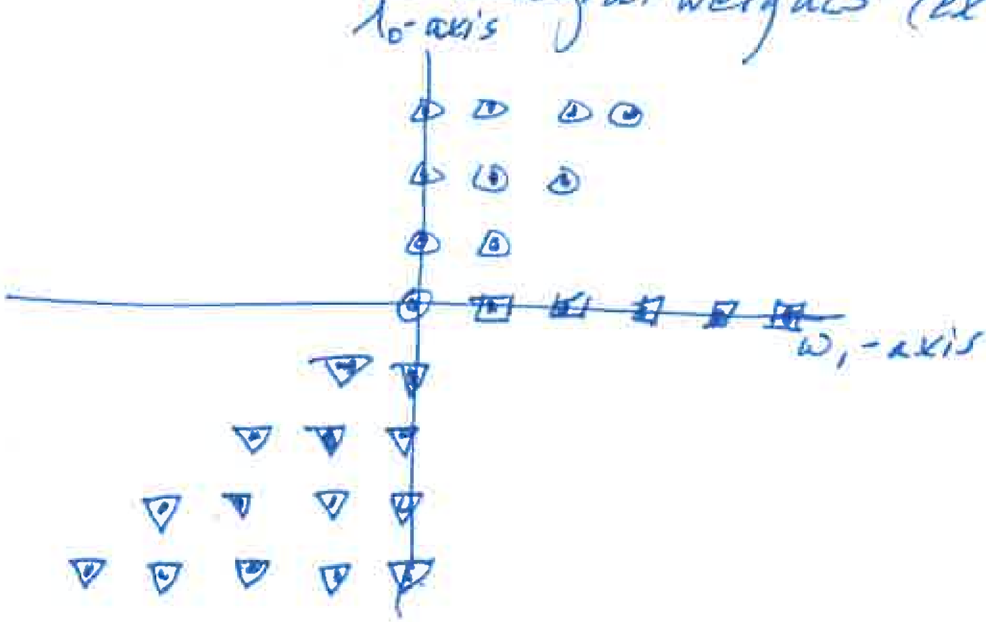
Where is the adjoint representation?

Integrable representations of $\mathfrak{g}$: Indexed by

(a) dominant integral weights (highest weight)

(b) antidominant integral weights (lowest weight)

(c) level 0 dominant integral weights (extremal weights)



Crystals: Definitions of Naito-Sagaki

Semiinfinite length for $W = \{ w t_{\mu^\vee} \mid w \in W_0,\ \mu^\vee \in \mathbb{Z}\text{-span}\{\alpha_1^\vee, \ldots$

$$\ell^{\frac{\infty}{2}}(w t_{\mu^\vee}) = \ell(w) + 2\langle \rho, \mu^\vee \rangle$$

The semiinfinite Bruhat graph has Vertices: W

directed labeled edges: $x \xrightarrow{\beta} s_\beta x$

if $\ell^{\frac{\infty}{2}}(s_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1$.

Fix λ (dominant integral, level 0).

Fix $a \in \mathbb{Q}$ with $0 < a < 1$.

Define $BG_a^{\frac{\infty}{2}}(\lambda)$: graph with vertices W

directed labeled edges: $x \xrightarrow{\beta} s_\beta x$

if $\ell^{\frac{\infty}{2}}(s_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1$ and $\langle a\lambda, \beta^\vee \rangle \in \mathbb{Z}$.

A semi-infinite LS path of shape λ is

$(x_1 \xleftarrow{a_1} x_2 \xleftarrow{a_2} \cdots \xleftarrow{a_{s-1}} x_s)$ with $0 < a_1 < \cdots < a_s$, $a_1, \ldots, a_s \in \mathbb{Q}$.

and there is a path $x_{i+1} \overset{a_i}{\rightsquigarrow} x_i$ in $BG_{a_i}^{\frac{\infty}{2}}(\lambda)$

The crystal of $L(\lambda)$ is

$$\mathbb{B}^{\frac{\infty}{2}}(\lambda) = \left\{ \begin{array}{c} \text{semiinfinite paths of} \\ \text{shape } \lambda \end{array} \right\}$$

Example $\mathring{\mathfrak{g}} = \mathfrak{sl}_2$. $W = \langle s_0, s_1 \mid s_0^2 = 1,\ s_1^2 = 1 \rangle$.

$t_{-\alpha_1}s_1 \quad t_{-\alpha_1} \qquad\qquad t_{\alpha_1}s_1 \quad t_{\alpha_1}$

$s_1s_0s_1 \quad s_1s_0 \quad s_1 \quad 1 \quad s_0 \quad s_0s_1 \quad s_0s_1s_0$

$-6 \quad 5 \quad -4 \quad 3 \quad -2 \quad 1 \quad 0 \quad -1 \quad 2 \quad -3 \quad 4 \quad -5 \quad 6$

The semi infinite Bruhat graph $BG^{\frac{\infty}{2}}$

$$\cdots \xrightarrow{\alpha_1^\vee} s_0s_1s_0 \xrightarrow{\alpha_0^\vee} s_1s_0 \xrightarrow{\alpha_1^\vee} s_0 \xrightarrow{\alpha_0^\vee} 1 \xrightarrow{\alpha_1^\vee} s_1 \xrightarrow{\alpha_0^\vee} s_0s_1 \xrightarrow{\alpha_1^\vee} s_1s_0s_1 \xrightarrow{\alpha_0^\vee} \cdots$$

~~Since~~ Let $\lambda = 2\omega_1$ and $a \in \mathbb{Q}$

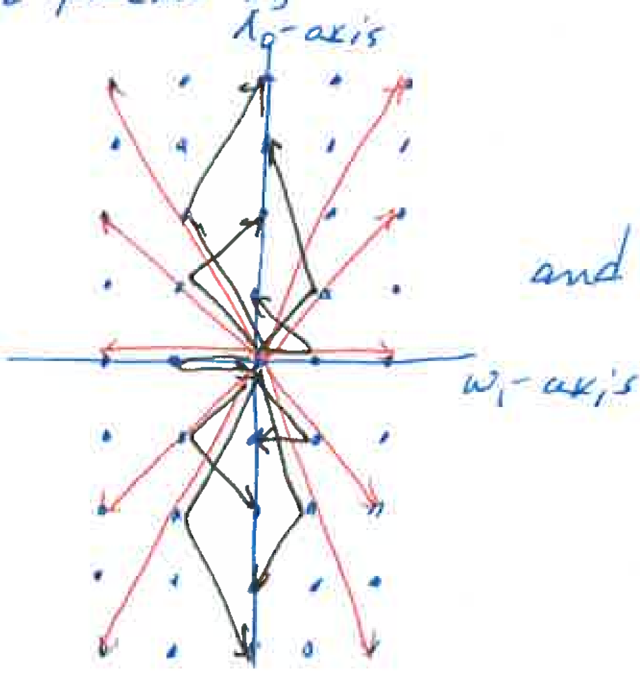
$$a\langle \lambda, \alpha_0^\vee \rangle = a\langle 2\omega_1, -\alpha_1^\vee + K \rangle = -2a$$

$$a\langle \lambda, \alpha_1^\vee \rangle = a\langle 2\omega_1, \alpha_1^\vee \rangle = 2a$$

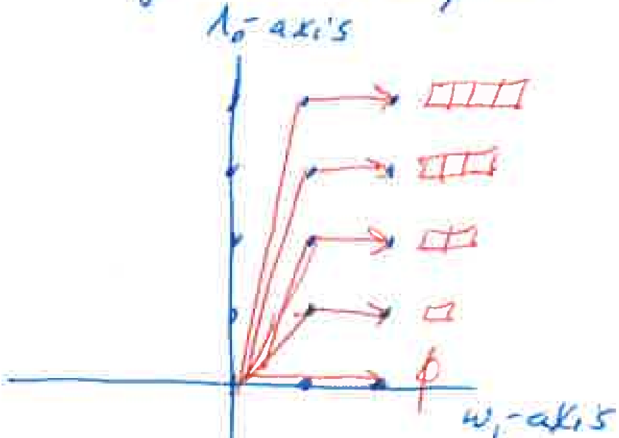
Thus

$$\mathbb{B}^{\frac{\infty}{2}}(2\omega_1) = \{x \mid x \in W\} \sqcup \left\{ x_2 t^{i}_{\varpi} x_1 \;\middle|\; \begin{array}{l} x_1, x_2 \in W \\ \text{and there is a path} \\ x_1 \text{ to } x_2 \text{ in } BG^{\frac{\infty}{2}} \end{array} \right\}$$

The connected component containing the straight line paths is



and



are the generating paths for the connected components of $\mathbb{B}^{\frac{\infty}{2}}(2\omega_1)$