The triumvirate of affine flag varieties

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Flags, Galleries and Reflections conference Sydney 5-8 August

Three flag varieties

Let $\mathring{G}(\mathbb{C}) = GL_n(\mathbb{C})$ $\quad G = \mathring{G}(\mathbb{C}((\epsilon))) = GL_n(\mathbb{C}((\epsilon)))$

Let $U^+(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \middle| * \in \mathbb{C} \right\}$ $\quad U^-(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \middle| * \in \mathbb{C} \right\}$

$= \langle x_\alpha(c) \mid c \in \mathbb{C}, \alpha \in \mathring{R}^+ \rangle$ $\quad = \langle x_{-\alpha}(c) \mid c \in \mathbb{C}, \alpha \in \mathring{R}^+ \rangle$

$D(\mathbb{C}^\times) = \left\{ \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \middle| \begin{matrix} d_i \in \mathbb{C}^\times \\ i \in \{1, \ldots, n\} \end{matrix} \right\} = \left\langle h_{\alpha_i^\vee}(d) \middle| \begin{matrix} d \in \mathbb{C}^\times \\ i \in \{0, \ldots, n\} \end{matrix} \right\rangle$

Define

$I^+ = \{ g \in G \mid g(0) \text{ exists and } g(0) \in U^+(\mathbb{C}) D(\mathbb{C}^\times) \}$

$I^- = \{ g \in G \mid g(\infty) \text{ exists and } g(\infty) \in U^-(\mathbb{C}) D(\mathbb{C}^\times) \}$

$I^0 = U^-(\mathbb{C}((\epsilon))) D(\mathbb{C}[[\epsilon]]^\times)$

Then

G/I^+ is the pos. level affine flag variety

G/I^0 is the level 0 affine flag variety

G/I^- is the neg. level affine flag variety

Three Bruhat decompositions

Let W be the affine Weyl group.

$$G = \bigsqcup_{w\in W} I^+ w I^+ \qquad \text{(pos. level)}$$

$$G = \bigsqcup_{v\in W} I^+ v I^0 \qquad \text{(level 0)}$$

$$G = \bigsqcup_{z\in W} I^+ z I^- \qquad \text{(neg. level)}$$

Three Bruhat orders

$x \le^+ w$ if $I^+ x I^+ \subseteq \overline{I^+ w I^+}$

$x \le^0 w$ if $I^+ x I^0 \subseteq \overline{I^+ w I^0}$

$x \le^- w$ if $I^+ x I^- \subseteq \overline{I^+ w I^-}$

Theorem Let $x, w \in W$ and let

$w = s_{j_1} \cdots s_{j_\ell}$ be a reduced word for w.

(a) $x \le^+ w \iff$ there is a reduced subword $x = s_{j_{i_1}} \cdots s_{j_{i_r}}$ of $w = s_{j_1} \cdots s_{j_n}$

(b) $x \le^- w \iff x \ge^+ w$

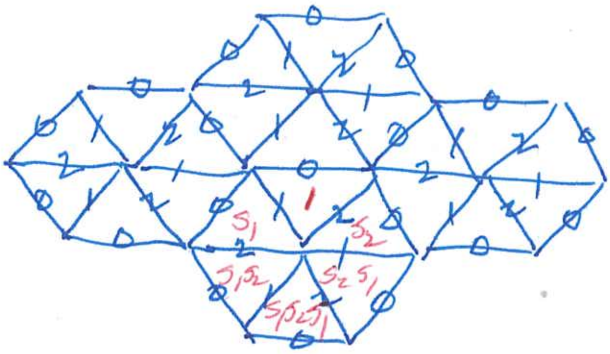
(c) (c1) If $\mu^\vee \in \mathfrak{h}_{\mathbb{Z}}$ then

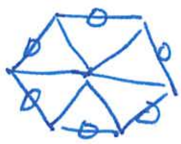
$$x \le^0 w \iff x t_{\mu^\vee} \le^0 w t_{\mu^\vee} \quad \text{(periodicity)}$$

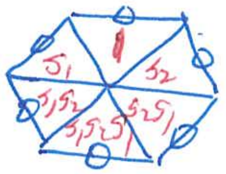
(c2) If x and w are dominant then

$$x \le^0 w \iff x \le^+ w \quad \text{(dominance)}.$$

the affine Weyl group



$W = \{\text{triangles}\}$ $\qquad \mathfrak{a}_{\mathbb{Z}} = \{$ hexagons  $\}$

$W_0 = \{$ triangles in identity hexagon  $\}$

$W = W_0 \ltimes \mathfrak{a}_{\mathbb{Z}}$

07.08.2019
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Labeling points of affine flag varieties

G is generated by SL_2s

$$\varphi_\alpha : SL_2(\mathbb{C}) \longrightarrow G$$
$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \longmapsto x_\alpha(c)$$
$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \longmapsto x_{-\alpha}(c)$$
$$\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \longmapsto h_{\alpha^\vee}(c)$$

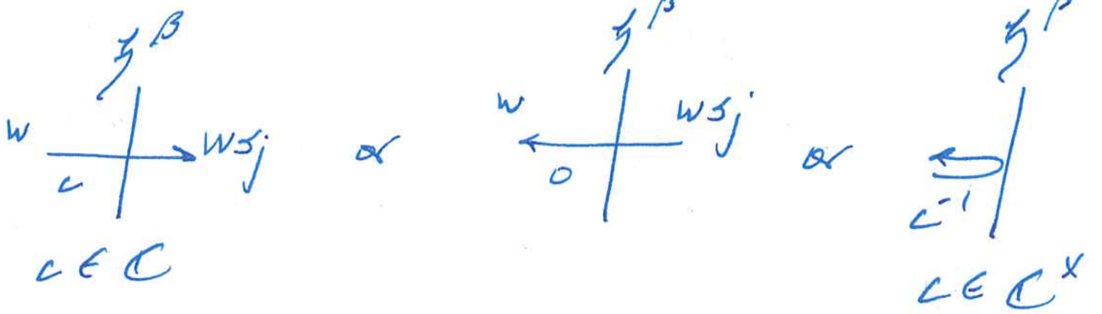
Note

$$x_{\alpha^\vee + k\delta}(c) = x_\alpha(c\epsilon^k) \quad (\text{relates } R^+ \text{ and } \mathring{R}^+)$$

Let

$$\varphi_{\alpha_i}\begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix} = y_i(c)$$

A blue / red / green labeled step of type j is



$c \in \mathbb{C}$ $\qquad$ $c \in \mathbb{C}^\times$

$w \le^+ ws_j$

$w \le^0 ws_j$

$w \le^- ws_j$

07.08.2019
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Let $w \in W$ and $v \in W$,

$w = s_{j_1} \cdots s_{j_\ell}$ a reduced word.

<u>Theorem</u> Let

$g \in I^+ w I^+ \cap I^+ v I^+$ (pos. level)

$g \in I^+ w I^+ \cap I^+ v I^0$ (level 0)

$g \in I^+ w I^+ \cap I^+ v I^-$ (neg. level)

Then there exists a <u>unique</u> blue / red / green labeled path

$p = (p_1, \ldots, p_\ell)$ of type $(j_1, \ldots, j_\ell)$ ending in v

and $b \in I^+$ such that

labels

$$g = \underbrace{x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell)}_{\text{steps of } p} \underbrace{y_{j_{i_1}}(0) \cdots y_{j_{i_r}}(0)}_{\text{crossings of } p} b$$

$$= x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) v b$$

$$\in x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) v b.$$

Hecke algebra actions

$$C(I^+\backslash G/I^+) = \mathrm{span}\{T_w \mid w \in W\}$$

$$C(I^0\backslash G/I^+) = \mathrm{span}\{X^v \mid v \in W\}$$

$$C(I^-\backslash G/I^+) = \mathrm{span}\{L_z \mid z \in W\}$$

Where
$$T_w = I^+ w I^+ = \sum_{g \in I^+ w I^+} g$$

$$X^v = I^+ v I \quad \text{and} \quad L_z = I^- z I^+$$

Replace $\mathbb{C}$ by $\mathbb{F}_q$ so that $G = \hat{G}(\mathbb{F}_q((\epsilon)))$

Then $H = C(I^+\backslash G/I^+)$ (the affine Hecke algebra)

acts by

$$T_w T_{s_j} = \begin{cases} T_{ws_j}, & \text{if } ws_j >_+ w, \\ q T_{ws_j} + (q-1) T_w, & \text{if } ws_j <_+ w. \end{cases}$$

$$X^v T_{s_j} = \begin{cases} X^{vs_j}, & \text{if } ws_j >_0 w, \\ q X^{vs_j} + (q-1) X^v, & \text{if } ws_j <_0 w, \end{cases}$$

$$L_z T_{s_j} = \begin{cases} L_{zs_j}, & \text{if } zs_j >_- z, \\ q L_{zs_j} + (q-1) L_z, & \text{if } zs_j <_- z. \end{cases}$$