Buildings

What is a building? Philosophy

(a) The building is a geometric object that the group acts on.

(b) The building is a simplicial complex and a metric space

(c) The building is G/B.

<u>One definition</u> (VERY useful, not perfect)

Let G be a group (any group)

B a subgroup

W an index set for B-double cosets

$$G = \bigsqcup_{w \in W} BwB$$

The building is

$$\mathcal{B} = G/B$$

with $\delta: \mathcal{B} \times \mathcal{B} \longrightarrow W$ given by

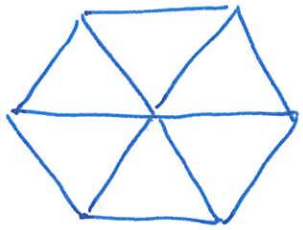
$$\delta(g_1B, g_2B) = w \quad \text{if} \quad Bg_1^{-1}g_2B = BwB.$$

Let me draw a picture

Example $W_{fin} = \langle s_1, s_2 \mid s_1^2 = 1,\ s_2^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$

$$= \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$$

acts on the Coxeter complex, a hexagon



Let $G = W$ and $B = \{1\}$.

One triangle for each coset gB

One blue edge for each coset gP_1 where $P_1 = \{1, s_1\}$

One red edge for each coset gP_2 where $P_2 = \{1, s_2\}$

One vertex for each coset gP where $P = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

The "building relation" here is the Coxeter relation

$$s_1 s_2 s_1 = s_2 s_1 s_2.$$

<u>Example</u> Glass Bead game 47:00

$$G = GL_3(\mathbb{F}_2) \quad \text{with} \quad B = B(\mathbb{F}_2)$$

$$\mathbb{F}_2 = \{0,1\}, \quad B = \left\{ \begin{pmatrix} x_1 & a_{12} & a_{13} \\ 0 & x_2 & a_{23} \\ 0 & 0 & x_3 \end{pmatrix} \in GL_3(\mathbb{F}_2) \right\}$$

Let $y_1(c) = \begin{pmatrix} c & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad y_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & -1 & 0 \end{pmatrix}$

$$s_1 = y_1(0) \qquad s_2 = y_2(0)$$

Then

$$G = B \sqcup Bs_1B \sqcup Bs_2B \sqcup Bs_1s_2B \sqcup Bs_2s_1B \sqcup Bs_1s_2s_1B$$

with

$$Bs_1B = \{ y_1(c)B \mid c \in \mathbb{F}_2 \}$$

$$Bs_2B = \{ y_2(c)B \mid c \in \mathbb{F}_2 \}$$

$$Bs_1s_2B = \{ y_1(c_1)y_2(c_2)B \mid c_1, c_2 \in \mathbb{F}_2 \}$$

$$Bs_2s_1B = \{ y_2(c_1)y_1(c_2)B \mid c_1, c_2 \in \mathbb{F}_2 \}$$

$$Bs_1s_2s_1B = \{ y_1(c_1)y_2(c_2)y_1(c_3)B \mid c_1, c_2, c_3 \in \mathbb{F}_2 \}$$

The building relation for this case:

$$y_1(c_1)y_2(c_2)y_1(c_3) = y_2(c_3)y_1(c_1c_3 - c_2)y_2(c_1)$$

The building has

One triangle for each coset gB

One blue edge for each coset gP_1

where $P_1 = B \sqcup B s_1 B$

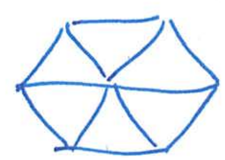
One red edge for each coset gP_2

where $P_2 = B \sqcup B s_2 B$

One vertex for each coset gP

where $P = G$.

Now we have <u>apartments</u>

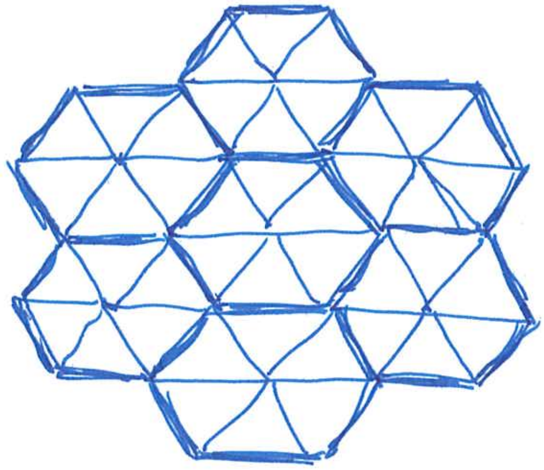
(each apartment is isomorphic to the Coxeter complex of W_{fin} )

For $c_1, c_2, c_3 \in \mathbb{F}_q$ the appartment is

$$A_{c_1,c_2,c_3} = \left\{ \begin{array}{cc} & B \\ y_1(c_1)B, & y_2(c_3)B, \\ y_1(c_1)y_2(c_2)B, & y_2(c_3)y_1(c_1c_3-c_2)B, \\ & y_1(c_1)y_2(c_2)y_1(c_3)B \end{array} \right\}$$

Example "The affine Coxeter complex"

$$W_{af} = \langle s_0, s_1, s_2 \mid s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i \in \mathbb{Z}/3\mathbb{Z} \rangle$$



$B = \{1\}$.

Example "The $\tilde{A}_2$ building"

$$G = GL_3(\mathbb{F}_2((\epsilon)))$$

$$B = I = \left\{ g \in GL_3(\mathbb{F}_2((\epsilon))) \;\middle|\; \begin{array}{l} g(0) \text{ exists and} \\ g(0) \in \mathbb{B}(\mathbb{F}_2) \end{array} \right\}$$

Then
$$G = \bigsqcup_{w \in W_{af}} IwI \quad \text{where}$$

if $w = s_{i_1} \cdots s_{i_\ell}$ is reduced then

$$I s_{i_1} \cdots s_{i_\ell} I = \{ y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) I \mid c_1, \ldots, c_\ell \in \mathbb{F}_2 \}$$

with

$$y_1(c) = \begin{pmatrix} c & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad y_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad y_0(c) = \begin{pmatrix} c & 0 & -\epsilon^{-1} \\ 0 & 1 & 0 \\ \epsilon & 0 & 0 \end{pmatrix}$$

for $c \in \mathbb{F}_2$.

Regions in $\mathcal{B}$ correspond to useful subsets of G

Let $\mathcal{B}$ be the building

c a favorite chamber

$\mathfrak{h}$ a fixed apartment containing c

Automorphisms and Stabilizers (In the good case)

$G = \mathrm{Aut}(\mathcal{B})$

$B = \{g \in G \mid gc = c\}$ "Borel subgroups"

$N = \{g \in G \mid g\mathfrak{h} = \mathfrak{h}\}$ "normalizer of H"

$H = \{g \in G \mid g \text{ fixes } \mathfrak{h} \text{ pointwise}\}$ "Cartan subgroup"

Groups like $G = GL_3(\mathbb{F}_2((\epsilon)))$ have two different buildings (which do talk to each other)

$\mathcal{B} = G/B = \{\text{"Borel subgroups" of } G\}$

$\mathcal{I} = G/I = \{\text{"Iwahori subgroups" of } G\}$.

By taking stabilizers,

simplices in $\mathcal{B}$ / $\mathcal{I}$ $\longleftrightarrow$ parabolic / parahoric subgroups of G

chambers in $\mathcal{B}$ / $\mathcal{I}$ $\longleftrightarrow$ minimal parabolic / parahoric subgroups of G

vertices in $\mathcal{B}$ / $\mathcal{I}$ $\longleftrightarrow$ maximal parabolic / parahoric subgroups of G

apartments in $\mathcal{I}$ $\longleftrightarrow$ maximal split tori in G

sectors in $\mathcal{I}$ $\longleftrightarrow$ parabolics in G.

For $\alpha \in \mathring{R}$,

the hyperplane $\mathfrak{h}^\alpha \longleftrightarrow \mathcal{U}_\alpha$ where

$\mathcal{U}_\alpha$ is the filtered sequence of groups

$$\cdots \supseteq \mathcal{U}_{\alpha-2\delta} \supseteq \mathcal{U}_{\alpha-\delta} \supseteq \mathcal{U}_{\alpha+0\delta} \supseteq \mathcal{U}_{\alpha+\delta} \supseteq \mathcal{U}_{\alpha+2\delta} \supseteq \cdots$$

where

$$\mathcal{U}_{\alpha+k\delta} = \{ x_\alpha(f) \mid f \in \epsilon^k \mathbb{F}_q[[\epsilon]] \}$$

The quotient in this sequence are the

<u>strips</u> parallel to $\mathfrak{h}^\alpha$ in $\mathcal{I}$.

<u>One official definition</u>

Let (W,S) be a Coxeter group.

A <u>chamber system</u> $\mathcal{B}$ on a set $S=\{s_1,\ldots,s_n\}$ is a set $\mathcal{B}$ with given equivalence relations $\sim_j$ for $j\in\{1,\ldots,n\}$.

A <u>gallery</u> of type $j_1,\ldots,j_\ell$ is a sequence
$$c_1\sim_{j_1}c_2,\quad c_2\sim_{j_2}c_3,\quad \ldots,\quad c_\ell\sim_{j_\ell}c_{\ell+1}$$

A <u>building of type (W,S)</u> is a chamber system $\mathcal{B}$ over S with a function $\delta: \mathcal{B}\times\mathcal{B}\to W$ such that

(B1) If $s_i\in S$ and $c\in\mathcal{B}$ then there exists $c'\in\mathcal{B}$ with $c'\neq c$ and $c'\sim_i c$.

(B2) If $w=s_{i_1}\cdots s_{i_\ell}$ is reduced and there is a gallery of type $j_1,\ldots,j_\ell$ from c to d then $\delta(c,d)=w$.

Example: "The affine Coxeter complex" $\tilde{A}_1$

$W_{af} = \langle s_0, s_1 \mid s_i^2 = 1 \rangle$ an infinite Dihedral group.

$B = \{1\}$

One edge $\overset{g}{\text{—}}$ for each coset gB

One blue vertex $\bullet$ for each coset gP_1
where $P_1 = \{1, s_1\}$

One red vertex $|$ for each coset gP_0
where $P_0 = \{1, s_0\}$

<u>Example</u> "The $\hat{A}_1$ building".

$$G = GL_2(\mathbb{F}_2((\varepsilon)))$$

$$B = I = \left\{ g \in GL_2(\mathbb{F}_2((\varepsilon))) \;\middle|\; \begin{array}{l} g(0) \text{ exists and} \\ g(0) \in B(\mathbb{F}_2) \end{array} \right\}$$

Then $G = \bigsqcup_{w \in W_{af}} IwI$ with

$$I s_{i_1} \cdots s_{i_\ell} I = \{ y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) I \mid c_1, c_2, \ldots, c_\ell \in \mathbb{F}_2 \}$$

where $y_1(c) = \begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix}$ and $y_0(c) = \begin{pmatrix} c & -\varepsilon^{-1} \\ \varepsilon & 0 \end{pmatrix}$

for $c \in \mathbb{F}_2$