Level 0 and the Bethe ansatz
Commuting elements

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Transfer matrices $t_V(u)$ are commuting elements in the quantum affine algebra U.

Analogy: Murphy elements are commuting elements in the group algebra of S_n

Representations are indexed by

$$\lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u)), \text{ with } \lambda_i(u) \in \mathbb{C}[u].$$

Analogy Irreducible representations of S_n are indexed by partitions with n boxes.

Eigenvalues are indexed by elements of $B(\lambda)$

Analogy Eigenvalues are contents of boxes in Standard Young tableaux.

Eigenvectors are constructed, inductively,

$$v_{f_i \gamma} = x_i(c) v_\gamma.$$

Analogy: Eigenvectors are given by

$$v_{s_i T} = \left(s_i - \frac{1}{c(T(i+1)) - c(T(i))}\right) v_T.$$

<u>Algebraic Bethe ansatz</u> (following Takhtajan-Faddeev 1979).

Let V and A be level 0 integrable U-modules.

$\rho : U \to \mathrm{End}(V)$ and $\pi : U \to \mathrm{End}(A)$ and $N \in \mathbb{Z}_{>0}$.

<u>Hamiltonian</u> $H_N : V^{\otimes N} \to V^{\otimes N}$ $\qquad H_N = \frac{d}{d\lambda}\left(\log(T_N(\lambda))\right)\Big|_{\lambda=\eta}$.

<u>Transfer matrix</u> $T_N(\lambda) : V^{\otimes N} \to V^{\otimes N}$ $\qquad T_N(\lambda) = Tr_A(\mathcal{T}_N(\lambda))$

<u>Monodromy matrix</u> $\mathcal{T}_N(\lambda) : V^{\otimes N} \otimes A \to V^{\otimes N} \otimes A$ $\qquad \mathcal{T}_N(\lambda) = (\rho^{\otimes N} \otimes \pi)(\mathcal{R}(\lambda))$

<u>R-matrix</u> $\mathcal{R}(\lambda) \in U \otimes U$ $\qquad \mathcal{R}(\lambda) = (id \otimes \tau_\lambda)(\mathcal{R})$

$(U, \mathcal{R}, \tau_\lambda)$ is a pseudoquasitriangular Hopf algebra

Examples of qKZB-algebras

rational Yangian $Y(\mathfrak{g})$ $= U_{XXX} = G_a(\mathfrak{g})$

trigonometric quantum affine algebra $U_q(L\mathfrak{g}) = U_{XXZ} = G_m(\mathfrak{g})$

elliptic elliptic quantum group $U_{XYZ} = G_\tau(\mathfrak{g})$

Degeneration

$$G_\tau(\mathfrak{g}) \underset{\tau\to i\infty}{\xrightarrow{\text{Tate curve}}} G_m(\mathfrak{g}) \underset{\text{term}}{\xrightarrow{\text{linear}}} G_a(\mathfrak{g})$$

In the special case $\mathfrak{g} = \mathfrak{sl}_2$, $V = \mathbb{C}^2$ and $A = \mathbb{C}^2$,

$$H_N = \sum_{i=1}^{N} J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z$$

where $\sigma^x, \sigma^y, \sigma^z$ are Pauli matrices, $\sigma_{N+1} = \sigma_1$ an.

$$\left.\begin{aligned} J_x &= 1 - k\,\mathrm{sn}_\tau^2(2\eta) \\ J_y &= 1 + k\,\mathrm{sn}_\tau^2(2\eta) \\ J_z &= \mathrm{cn}_\tau(2\eta)\,\mathrm{dn}_\tau(2\eta) \end{aligned}\right\}$$ In terms of Jacobi theta functions

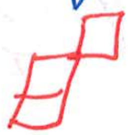
Eigenvalues/eigenvectors of $\Phi_i^+(u)$

Calibration graph

Vertices: γ, one for each eigenvector v_γ

Edges: $\gamma \xrightarrow{x_i(c)} \gamma'$, if $v_{\gamma'} = x_i(c) v_\gamma$

An example (across subfields)

- BFKL perfect crystal (Benkart-Frenkel-Kang-Lee)
- column strict tableaux shape 
- nonsymmetric Macdonald polynomial $E_{(0,1,2)}(q,0)$
- Nakajima's monomial crystal
  Frenkel-Reshetikhin / Frenkel-Hernandez

~~Conversion~~ Boxes to eigenvalues

$$\mathrm{wt}\left(\boxed{i}_{\,a}\right) = \frac{1-u_i a q^{i-1}}{1-u_{i-1} a q^{i}} \quad \text{and} \quad \gamma = \prod_{\text{boxes}} \mathrm{wt}(\text{box})$$

then

$$\Phi_i^+(u) v_\gamma = \frac{\gamma_i(uq^{-1})}{\gamma_i(uq)} v_\gamma \quad \text{where } \gamma_i(u) = \gamma(u_1,\dots,u_n)\Big|_{\substack{u_i = u \\ u_j = 0 \text{ for } j\neq i}}$$

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