Draft lecture for Rep. Thy seminar 17.03.2020 ①

Macdonald polynomials

DAHA $\tilde{H}$ has subalgebras

$\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ and H_0 and

$\tilde{H} = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \otimes H_0 \otimes \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ as vectorspaces.

Polynomial representation

$$\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] = \mathrm{Ind}_{H_0 \otimes \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]}^{\tilde{H}} (\mathrm{triv}).$$

Macdonald polynomials $E_\mu \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$

simultaneous eigenvectors for $Y_1, \ldots, Y_n$.

Intertwiners $\tau_0^\vee, \ldots, \tau_n^\vee$ such that

$Y^\gamma \tau_i^\vee = \tau_i^\vee Y^{s_i \gamma}$ where $Y^\gamma = Y_1^{\gamma_1} \cdots Y_n^{\gamma_n}$.

So

$$E_\mu = \tau_{m_\mu} E_0 = \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee E_0, \quad \text{with } E_0 = 1$$

and $m_\mu = s_{i_1} \cdots s_{i_\ell}$ a chosen reduced word for the minimal length element in $t_\mu W_0$ (a coset in W/W_0).

Let $u = t_\mu^{-1} m_\mu$

R-Yip formula

$$E_\mu = \sum_{p \in B(1, \vec{m}_\mu)} wt(p)\, X^{endpt(p)}, \qquad \text{where}$$

$B(1, \vec{m}_\mu) = \{$foldings p of the path $\vec{m}_\mu\}$

and

$$wt(p) = t^{\frac{1}{2}\ell(\varphi(p))} \left(\prod_{k \in f^+(p)} \frac{t^{-\frac{1}{2}}(1-t)}{1-q^{sh(\beta_k^\vee)} t^{ht(\beta_k^\vee)}} \right) \left(\prod_{k \in f^-(p)} \frac{t^{-\frac{1}{2}}(1-t)\, q^{sh(\beta_k^\vee)} t^{ht(\beta_k^\vee)}}{1-q^{sh(\beta_k^\vee)} t^{ht(\beta_k^\vee)}} \right)$$

HHL formula (for GL_n)

$$E_\mu = \sum_{T \in J_\mu} wt(T)\, x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots x_n^{\#n\text{'s in } T}$$

where

$J_\mu = \{$nonattacking fillings T of $\mu\}$

and

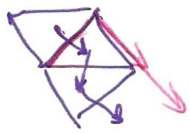
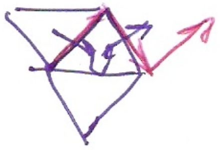
$$wt(T) = q^{maj(T)} t^{coinv(T)} \prod_{\substack{u \in T \\ T(u) \neq T(d(u))}} \frac{1-t}{1-q^{leg(u)+1} t^{arm(u)+1}}$$

An example

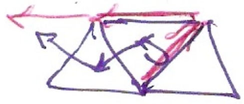
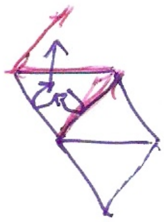
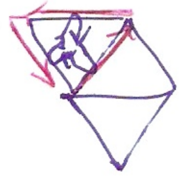
$$E_{(1,0,2)} = x_3x_3x_1 + \frac{1-t}{1-qt}x_1x_3x_1 + \frac{1-t}{1-qt}x_2x_3x_1$$

$$\begin{array}{ccc} & & 3\\ 1 & & 3\\ 1 & 2 & 3\end{array} \qquad \begin{array}{ccc} & & 1\\ 1 & & 3\\ 1 & 2 & 3\end{array} \qquad \begin{array}{ccc} & & 2\\ 1 & & 3\\ 1 & 2 & 3\end{array}$$

$E_{s_1 s_2 p} =$


$$+ \frac{(1-t)}{1-q^2t^2}x_2x_2x_1 + \frac{(1-t)(1-t)}{(1-q^2t^2)(1-qt)}x_1x_2x_1 + \frac{(1-t)(1-t)}{(1-q^2t^2)(1-qt)}q\,x_3x_2x_1$$

$f^+(p) = \{$steps which are positive folds of $p\}$

$f^-(p) = \{$steps which are negative folds of $p\}$

$f(p) = f^+(p) \sqcup f^-(p)$

$\varphi(p)$ is the "final direction" of p.

<u>Relation and specialization</u>

$$\mathrm{coinv}^{\pm}(p) = \left(\sum_{k \in f^{\pm}(p)} ht(\beta_k^{\vee})\right) + \frac{1}{2}\left(\ell(\varphi(p)) - \# f(p) - \ell(m)\right)$$

$$\mathrm{maj}^{\pm}(p) = \sum_{k \in f^{\pm}(p)} sh(\beta_k^{\vee})$$

Then $E_{\mu}(q,t)$

$$= \sum_{p \in B(1,\vec{m}_{\mu})} X^{\mathrm{endpt}(p)} q^{\mathrm{maj}^{-}(p)} t^{\mathrm{coinv}^{-}(p)} \prod_{k \in f(p)} \frac{1-t}{1-q^{sh(\beta_k^{\vee})} t^{ht(\beta_k^{\vee})}}$$

$$= \sum_{p \in B(1,\vec{m}_{\mu})} X^{\mathrm{endpt}(p)} q^{-\mathrm{maj}^{+}(p)} t^{-\mathrm{coinv}^{+}(p)} \prod_{k \in f(p)} \frac{1-t^{-1}}{1-q^{-sh(\beta_k^{\vee})} t^{-ht(\beta_k^{\vee})}}$$

p is <u>positively</u> / <u>negatively</u> <u>folded</u> if $\mathrm{maj}^{\mp}(p) = 0$

p is <u>positively</u> / <u>negatively</u> <u>semiinfinite</u> if $\mathrm{coinv}^{\mp}(p) = 0$.

Then

$$E_\mu(0,t) = \sum_{\substack{p \in B(l,\vec{m}_\mu) \\ p \text{ pos. folded}}} t^{\mathrm{coinv}^-(p)} (1-t)^{\#f(p)} X^{\mathrm{endpt}(p)}$$

$$E_\mu(q,0) = \sum_{\substack{p \in B(l,\vec{m}_\mu) \\ p \text{ neg. semiinf}}} q^{\mathrm{maj}^-(p)} X^{\mathrm{endpt}(p)}$$

$$E_\mu(\infty,t) = \sum_{p \text{ neg folded}} t^{\mathrm{coinv}^+(p)} (1-t^{-1})^{\#f(p)} X^{\mathrm{endpt}(p)}$$

$$E_\mu(q,\infty) = \sum_{p \text{ pos semiinf}} q^{-\mathrm{maj}^+(p)} X^{\mathrm{endpt}(p)}.$$

17.03.2020 ⑥

① Corteel-Mandelshtam-Williams, From multiline queues to Macdonald polynomials via the exclusion process

$$f_\lambda(x_1,\dots,x_n) = \sum_{\substack{\text{multiline queues}\\ \text{of type } \lambda}} wt(Q)$$

② Cantini-de Gier-Wheeler, Matrix product formula for Macdonald polynomials, arXiv:1505.00287

$$f_\lambda(x_1,\dots,x_n) = \frac{1}{\Omega_{\lambda^+}} \operatorname{tr}\left(A_{\lambda_1}(x_1)\cdots A_{\lambda_n}(x_n) S\right)$$

③ Kasatani-Takeyama, arXiv:math/0608773

As H-modules,

$$\mathbb{C}[X_1^{\pm 1},\dots,X_n^{\pm 1}] = \bigoplus_{\lambda \text{ partition}} H^\lambda$$

and H^λ has basis $\{ f_{w\lambda} \mid w \in S_n/Stab(\lambda)\}$ with

$f_\lambda = E_\lambda$ and $f_{s_i\mu} = T_i f_\mu$ if $\mu_i > \mu_{i+1}$.

$\boxed{3B}$ Kasatani-Takeyama, arXiv:math/0608773

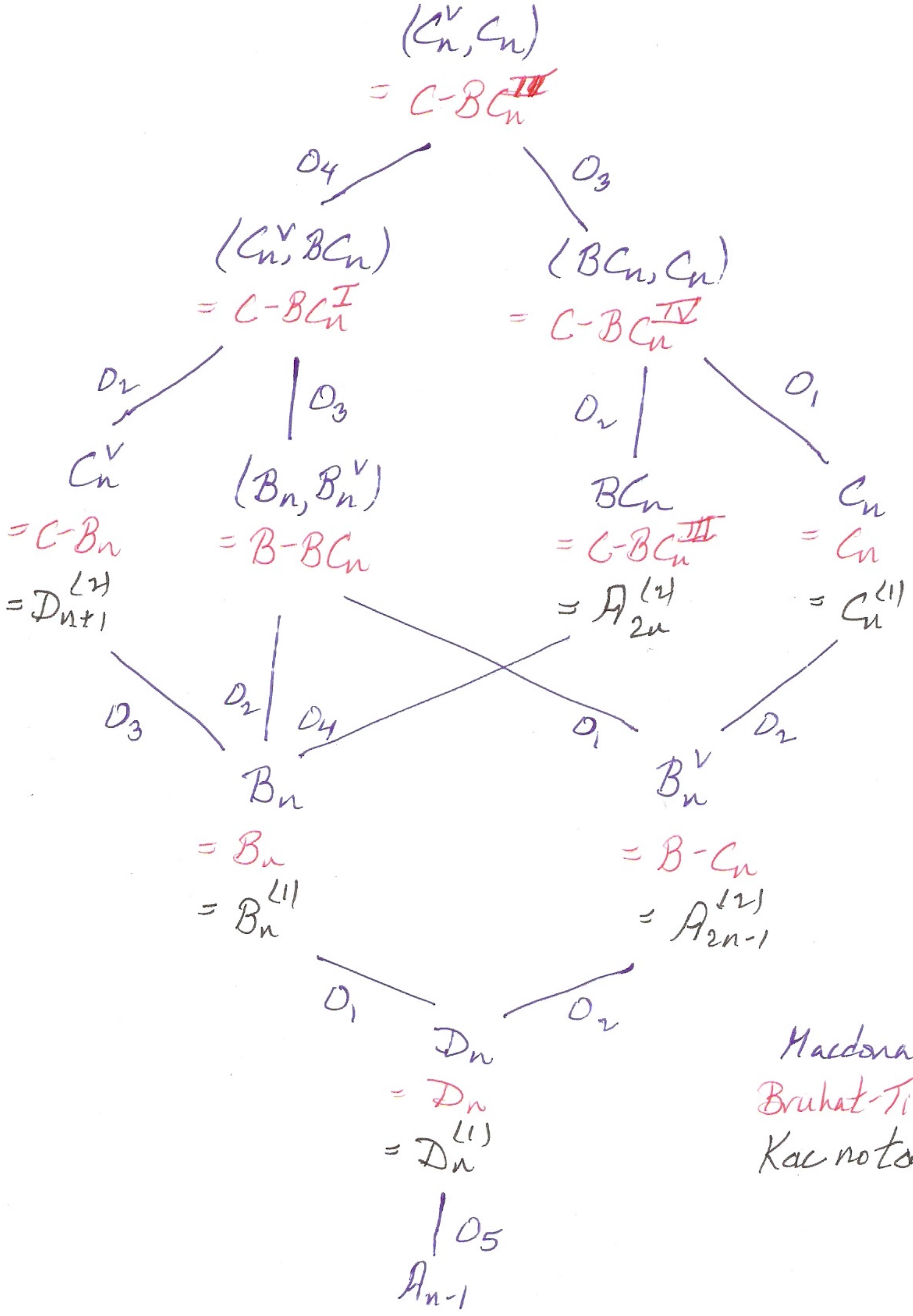
$$G = K \sum_{\mu \in S_n\lambda} f_\mu \, v_{\mu_1} \otimes \cdots \otimes v_{\mu_n} \in \left(\mathbb{C}^N[z^{\pm 1}]\right)^{\otimes n}$$

is a solution of qKZ equation.

(4) For each n, there are

5 finite root systems of classical type
A_n, B_n, C_n, D_n, BC_n

10 (yes ten!) or eleven affine root systems of classical type
(see Macdonald's 2003 book).



Macdonald notation
Bruhat-Tits notation
Kac notation