12.05.2020 ①
Talkgroup
Berkeley A. Ram

Macdonald polynomials

Nicolle said: $\mu \in \mathbb{Z}^n_{\geq 0}$ (the set of compositions)

$E_\mu \xrightarrow[t=0]{q=0} K_\mu$ Key polynomial
Demazure character

$E_\mu \xrightarrow[t=\infty]{q=\infty} A_\mu$ Demazure atom.

Example $E_{(2,0,1)}$

$$= x_1 x_3 x_1 + \frac{(1-t)}{(1-qt)} x_1 x_2 x_1 + \frac{(1-t)}{(1-qt^2)} qt\, x_1 x_3 x_2$$

$$+ \frac{(1-t)}{(1-qt)} \frac{(1-t)}{(1-qt^2)} q\, x_1 x_2 x_3$$

$$= x_1 x_3 x_1 + \frac{(1-t^{-1})}{(1-q^{-1}t^{-1})} q^{-1} x_1 x_2 x_1 + \frac{1-t^{-1}}{1-q^{-1}t^{-2}} x_1 x_3 x_2$$

$$+ \frac{(1-t^{-1})}{(1-q^{-1}t^{-1})} \frac{(1-t^{-1})}{(1-q^{-1}t^{-2})} q^{-1} t^{-1} x_1 x_2 x_3$$

Let λ be the increasing rearrangement of μ

$\mu = w\lambda$ with $w \in S_n$

Then
$$K_\mu = \pi_w(x^\lambda) \quad \text{with} \quad \pi_w = \pi_{i_1} \cdots \pi_{i_\ell}$$
$$A_\mu = \theta_w(x^\lambda) \qquad \theta_w = \theta_{i_1} \cdots \theta_{i_\ell}$$

with $w = s_{i_1} \cdots s_{i_\ell}$ reduced.

$$\pi_i : \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \longrightarrow \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$
$$\theta_i : \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \longrightarrow \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Intertwiners $\tau_i^\vee : \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Then
$$E_\mu = \tau_w^\vee(E_\lambda) \quad \text{with} \quad \tau_w^\vee = \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee.$$

~~Even~~ and $E_\lambda \xrightarrow[t=0]{q=0} x^\lambda$ and $\tau_i^\vee \xrightarrow[t=0]{q=0} \pi_i$

$E_\lambda \xrightarrow[t=\infty]{q=\infty} x^\lambda$ $\qquad \tau_i^\vee \xrightarrow[t=\infty]{q=\infty} \theta_i$.

Even better:

$$E_\mu = \tau_{m_\mu}^\vee(1) \quad \text{and} \quad E_0 = 1.$$

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What is m_μ?

W = affine Weyl group.

An *affine permutation* is a bijection

$$w : \mathbb{Z} \to \mathbb{Z} \quad \text{with } w(i+n) = w(i)+n.$$

Define

s_i switches i and $i+1$.

π given by $\pi(i) = i+1$.

$t_\mu = t_{(\mu_1, \ldots, \mu_n)}$ given by $t_\mu(i) = i - n\mu_i$.

Then

$m_\mu = t_\mu w_\mu$ where $w_\mu \in S_n$

is minimal length such that

$w_\mu \cdot \mu$ is increasing.

Examples $\mu = (0, 4, 1, 5, 4)$

$$m_\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1-5\cdot 0 & 3-5\cdot 4 & 2-5\cdot 1 & 5-5\cdot 5 & 4-5\cdot 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & -17 & -3 & -20 & -16 \end{pmatrix}$$

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Favorite reduced word for m_μ

W acts on $\mathbb{Z}^n$.

$$\pi^{-1}(\mu_1,\ldots,\mu_n) = (\mu_2,\ldots,\mu_n,\mu_1-1)$$

$$\pi(\mu_1,\ldots,\mu_n) = (\mu_n+1,\mu_1,\ldots,\mu_{n-1}).$$

Example $(2,0,1) \underset{\pi}{\overset{\pi^{-1}}{\rightleftarrows}} (0,1,1) \underset{s_1}{\overset{s_1}{\rightleftarrows}} (1,0,1) \underset{\pi}{\overset{\pi^{-1}}{\rightleftarrows}} (0,1,0)$

$\underset{s_1}{\overset{s_1}{\rightleftarrows}} (1,0,0) \underset{\pi}{\overset{\pi^{-1}}{\rightleftarrows}} (0,0,0)$

So

$$m_{(2,0,1)} = \pi s_1 \pi s_1 \pi$$

and

$$E_{(2,0,1)} = \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee \tau_1^\vee \cdot 1 = \tau_{m_\mu}^\vee \cdot 1$$

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Words

A <u>word</u> is a sequence $\vec{x} = x_{j_1} \cdots x_{j_\ell}$ of letters from the alphabet $\{x_1, \ldots, x_n\}$.

1 is the empty word.

Combinatorial intertwiners

$$\hat{\tau}_i^\vee : \left\{\begin{matrix}\text{sets of}\\ \text{words}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{sets of}\\ \text{words}\end{matrix}\right\}$$

$$\hat{\tau}_\pi^\vee (x_{j_1} \cdots x_{j_\ell}) = x_1 x_{j_1+1} \cdots x_{j_\ell+1} \quad \left(\begin{matrix}\text{indices}\\ \text{mod } n\end{matrix}\right)$$

$$\hat{\tau}_i^\vee (x_{j_1} \cdots x_{j_\ell}) = \{x_{s_i(j_1)} \cdots x_{s_i(j_\ell)},\ x_{j_1} \cdots x_{j_\ell}\}.$$

Then

$$E^{wd}_{(2,0,1)} = \hat{\tau}_\pi^\vee \hat{\tau}_1^\vee \hat{\tau}_\pi^\vee \hat{\tau}_1^\vee \hat{\tau}_\pi^\vee \{1\}$$

$$= \hat{\tau}_\pi^\vee \hat{\tau}_1^\vee \hat{\tau}_\pi^\vee \hat{\tau}_1^\vee \{x_1\} = \hat{\tau}_\pi^\vee \hat{\tau}_1^\vee \hat{\tau}_\pi^\vee \{x_2, x_1\}$$

$$= \hat{\tau}_\pi^\vee \hat{\tau}_1^\vee \{x_1 x_3, x_1 x_2\} = \hat{\tau}_\pi^\vee \{x_2 x_3, x_1 x_3, x_2 x_1, x_1 x_2\}$$

$$= \{x_1 x_3 x_1, x_1 x_2 x_1, x_1 x_3 x_2, x_1 x_2 x_3\}$$

$$\begin{array}{ccc} 1 & & \\ 1 & & 3 \\ \hline 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} 1 & & \\ 1 & & 2 \\ \hline 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} 2 & & \\ 1 & & 3 \\ \hline 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} 3 & & \\ 1 & & 2 \\ \hline 1 & 2 & 3 \end{array}$$

Bijection

$$E_\mu^{wd} \longleftrightarrow \left\{\begin{matrix}\text{nonattacking}\\ \text{fillings of}\\ \text{shape } \mu\end{matrix}\right\}$$

$$E_\mu = \sum_{\vec{x} \in \mathcal{E}_\mu^{wd}} wt(\vec{x})\, \vec{x}$$ (nonattacking fillings formula)

<u>Alcove walks</u> An <u>alcove walk</u> is a sequence

$\vec{v} = v_{j_1} \cdots v_{j_\ell} w_0$ with $v_{j_k} \in \{\pi^\vee, \underbrace{c_1, \ldots, c_{n-1}}_{\text{crossings}}, \underbrace{f_1, \ldots, f_{n-1}}_{\text{folds}}\}$

w_0 is the empty alcove walk.

$$\tau_i^\vee : \begin{Bmatrix}\text{sets of} \\ \text{alcove} \\ \text{walks}\end{Bmatrix} \longrightarrow \begin{Bmatrix}\text{sets of} \\ \text{alcove} \\ \text{walks}\end{Bmatrix}$$

$$\tau_\pi^\vee (v_{j_1} \cdots v_{j_\ell} w_0) = \pi^\vee v_{j_1} \cdots v_{j_\ell} w_0$$

$$\tau_i^\vee (v_{j_1} \cdots v_{j_\ell} w_0) = \{c_i v_{j_1} \cdots v_{j_\ell} w_0, f_i v_{j_1} \cdots v_{j_\ell} w_0\}$$

<u>Example</u> $\mathcal{E}^{aw}_{(2,0,1)} = \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee \{w_0\}$

$= \tau_\pi^\vee \tau_1^\vee \tau_\pi^\vee \{c_1 \pi^\vee w_0, f_1 \pi^\vee w_0\}$

$= \tau_\pi^\vee \{c_1 \pi^\vee c_1 \pi^\vee w_0, f_1 \pi^\vee c_1 \pi^\vee w_0, c_1 \pi^\vee f_1 \pi^\vee w_0, f_1 \pi^\vee f_1 \pi^\vee w_0\}$

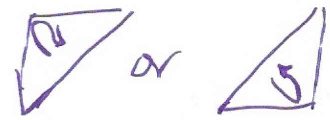
Then

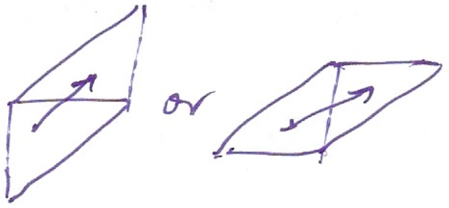
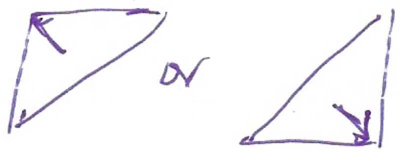
$$E_\mu = \sum_{\vec{v} \in \mathcal{E}_\mu^{aw}} wt(\vec{v})\, \cancel{x}^{ep(\vec{v})}$$ (alcove walk formula)

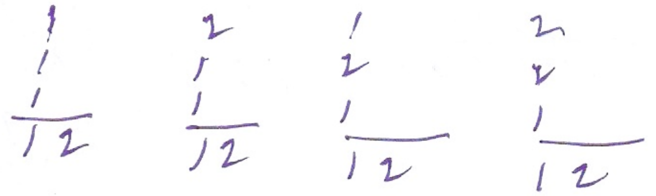
Paths

For GL_n paths are in $\mathbb{R}^n$.

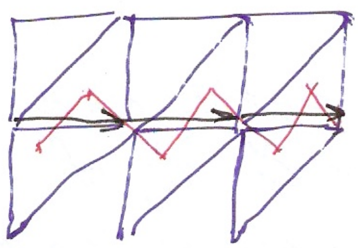
x_i is a path one unit in i-direction.

$e_i =$  $f_i =$ 

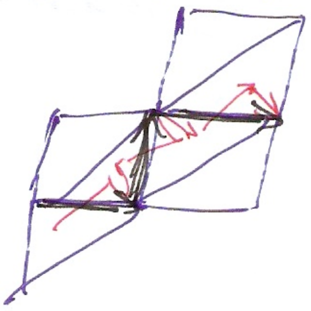
$\pi^\vee =$  $w_0 =$ 

Example $E_{(3,0)}$ 

$x_1x_1x_1$ $x_1x_1x_2$ $x_1x_2x_1$ $x_1x_2x_2$



$x_1x_1x_1$ "=" $\pi^\vee e_1 \pi^\vee e_1 \pi^\vee w_0$



$x_1x_2x_1$ "=" $\pi^\vee f_1 \pi^\vee f_1 \pi^\vee w_0$