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Berkeley Talk
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Formulas for Macdonald Polynomials

$\mathcal{E}_\mu^{wd} = \tilde{\tau}^\vee_{m_\mu}\{1\}$ the set of nonattacking fillings

$\mathcal{E}_\mu^{aw} = \tilde{\tau}^\vee_{m_\mu}\{w_0\}$ the set of alcove walks

The nonsymmetric Macdonald polynomial:

$$E_\mu = \sum_{\vec{x} \in \mathcal{E}_\mu^{wd}} wt_\mu(\vec{x})\, x^{ep(\vec{x})}$$ nonattacking fillings formula

$$E_\mu = \sum_{\vec{v} \in \mathcal{E}_\mu^{aw}} wt_\mu(\vec{v})\, x^{ep(\vec{v})}$$ alcove walk formula

Here
$$wt_\mu(\vec{x}) = t^{\#D(\vec{x}) - \Delta(\vec{x})}$$
$$\cdot \left(\prod_{s \in D(\vec{x})} \frac{1-t}{1-q^{leg(s)+1} t^{arm(s)+1}}\right)\left(\prod_{s \in A(\vec{x})} \frac{(1-t)\, q^{leg(s)+1} t^{arm(s)+1}}{1-q^{leg(s)+1} t^{arm(s)+1}}\right)$$

and
$$wt_\mu(\vec{v}) = t^{\frac{1}{2}(\ell(\varphi(\vec{v})) - \ell(w_\mu) - \#f(\vec{v}))}$$
$$\cdot \left(\prod_{\beta_k^\vee \in f^+(\vec{v})} \frac{1-t}{1-q^{sh(\beta_k^\vee)} t^{ht(\beta_k^\vee)}}\right)\left(\prod_{\beta_k^\vee \in f^-(\vec{v})} \frac{(1-t)\, q^{sh(\beta_k^\vee)} t^{ht(\beta_k^\vee)}}{1-q^{sh(\beta_k^\vee)} t^{ht(\beta_k^\vee)}}\right)$$

What is $\beta_k^\vee$?

<u>Inversions</u> W = affine Weyl group.

An <u>affine permutation</u> is a bijection

$$w: \mathbb{Z} \to \mathbb{Z} \quad \text{with} \quad w(i+n) = w(i)+n.$$

If $w \in W$ then $w = t_\mu v$ with $v \in S_n$, $\mu = (\mu_1, \ldots, \mu_n)$

$$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ v(1)-n\mu_1, & v(2)-n\mu_2 & \cdots & v(n)-n\mu_n \end{pmatrix}$$

An <u>inversion of w</u> is (j,k) with $j<k$ and $w(j) > w(k)$

<u>Example</u> $\mu = (0, 4, 1, 5, 4)$ $\qquad E_\mu = t_{m_\mu}^{\vee} \cdot 1.$

$$m_\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1-5\cdot 0 & 3-5\cdot 4 & 2-5\cdot 1 & 5-5\cdot 5 & 4-5\cdot 4 \end{pmatrix}$$

$$m_\mu^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1+5\cdot 0 & 3+5\cdot 1 & 2+5\cdot 4 & 5+5\cdot 4 & 4+5\cdot 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 1 & 8 & 22 & 25 & 29 & 6 & 13 & 27 & 30 & 34 & 11 & 18 & 32 & 35 & 39 & 16 & 23 & 37 \end{pmatrix} \cdots$$

Inversions of m_μ^{-1}:

(3,21)	(3,16)	(3,12) (3,11)	(3,7) (3,6)	
(2,6)				
(5,26)	(5,22) (5,21)	(5,17) (5,16)	(5,12) (5,11)	(5,8) (5,7) (5,6)
(4,21)	(4,17) (4,16)	(4,12) (4,11)	(4,7) (4,6)	

Convert inversions to affine roots

$\langle j,k\rangle = \langle j, i+\ell n\rangle = \ell K + \varepsilon_i^\vee - \varepsilon_j^\vee$ with $i,j\in\{1,\dots,n\}$, $\ell\in\mathbb{Z}_{\geq 0}$.

$4K+\varepsilon_1^\vee-\varepsilon_3^\vee$	$3K+\varepsilon_2^\vee-\varepsilon_3$ $3K+\varepsilon_1^\vee-\varepsilon_3^\vee$	$2K+\varepsilon_2^\vee-\varepsilon_3^\vee$ $2K+\varepsilon_1^\vee-\varepsilon_3^\vee$	$K+\varepsilon_2^\vee-\varepsilon_3^\vee$ $K+\varepsilon_1^\vee-\varepsilon_3^\vee$	
$K+\varepsilon_1^\vee-\varepsilon_2^\vee$				
$5K+\varepsilon_1^\vee-\varepsilon_5^\vee$	$4K+\varepsilon_2^\vee-\varepsilon_5^\vee$ $4K+\varepsilon_1^\vee-\varepsilon_5^\vee$	$3K+\varepsilon_2^\vee-\varepsilon_5^\vee$ $3K+\varepsilon_1^\vee-\varepsilon_5^\vee$	$2K+\varepsilon_2^\vee-\varepsilon_5^\vee$ $2K+\varepsilon_1^\vee-\varepsilon_5^\vee$	$K+\varepsilon_3^\vee-\varepsilon_5^\vee$ $K+\varepsilon_2^\vee-\varepsilon_5^\vee$ $K+\varepsilon_1^\vee-\varepsilon_5^\vee$
$4K+\varepsilon_1^\vee-\varepsilon_4^\vee$	$3K+\varepsilon_2^\vee-\varepsilon_4^\vee$ $3K+\varepsilon_1^\vee-\varepsilon_4^\vee$	$2K+\varepsilon_2^\vee-\varepsilon_4^\vee$ $2K+\varepsilon_1^\vee-\varepsilon_4^\vee$	$K+\varepsilon_2^\vee-\varepsilon_4^\vee$ $K+\varepsilon_1^\vee-\varepsilon_4^\vee$	

Shift and height

$$\mathrm{sh}(\ell K+\varepsilon_i^\vee-\varepsilon_j^\vee)=\ell \quad\text{and}\quad \mathrm{ht}(\ell K+\varepsilon_i^\vee-\varepsilon_j^\vee)=j-i.$$

The shift and height of the top root in each box:

q^4t^2	q^3t	q^2t	qt	
qt				
q^5t^4	q^4t^3	q^3t^3	q^2t^3	qt^2
q^4t^3	q^3t^2	q^2t^2	qt^2	

$$(\mathrm{sh}(\beta_k^\vee), \mathrm{ht}(\beta_k^\vee)) = (\mathrm{leg}(s)+1, \mathrm{arm}(s)+1).$$

My Philosophies

(1) Everything comes from DAHA which has

$$\underbrace{T_1, \ldots, T_{n-1}}_{\text{Demazure-Lusztig operators}}, \quad \underbrace{X_1, \ldots, X_n}_{\text{multiplication by } X_i} \quad \underbrace{Y_1, \ldots, Y_n}_{\text{Cherednik-Dunkl operators}}$$

Intertwiners: $E_\mu = \tau^\vee_{m_\mu} 1$, $\tau^\vee_{m_\mu} = \tau^\vee_{i_1} \cdots \tau^\vee_{i_\ell}$ and

$$\tau^\vee_i = T_i + \frac{t^{\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}} = T_i^{-1} + \frac{t^{-\frac{1}{2}}(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}}.$$

(2) Symmetrization (Macdonald Sém Bourbaki 1995)

$$P_\lambda = 1_0 E_\mu = \sum_{w \in W_{fin}} t^{-\frac{1}{2}\ell(w_0 w)} T_v E_\mu \quad \text{and}$$

$E^v_\mu = t^{-\frac{1}{2}\ell(w_0 v)} T_v E_\mu$ are the permuted basement Macdonald polynomials of Fereira and Alexandersson.

(3) Standard variables for $\mathbb{C}[X_1^{\pm 1}, \ldots X_n^{\pm 1}]$:

x_i orthogonal basis — nonattack fillings formula -

$X^{\alpha_i} = x_i x_{i+1}^{-1}$ simple roots — alcove walk formula.

$X^{\omega_i} = x_1 \cdots x_i$ fundamental weights — one-skeleton formula $\overset{?}{=}$ HLQ or Lenart.

(4) For each n, there are

5 finite root systems of classical type

$A_n,\ B_n,\ C_n,\ D_n,\ BC_n$

10 (yes ten!) affine root systems of classical type
or eleven

(see Macdonald's 2003 book).