Flags, Crystals and Orthogonal polynomials 15.10.2020 (1)

Loop groups $L\mathring{G}$

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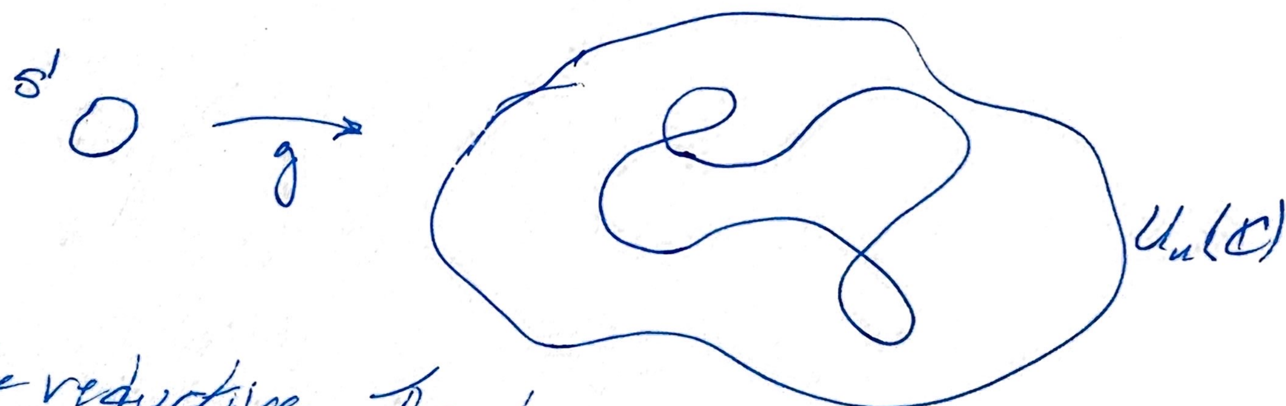
$$GL_1(\mathbb{C}) = \mathbb{C}^\times \xrightarrow{g} GL_n(\mathbb{C})$$
$$t \longmapsto (g_{ij}(t))$$

complexification maximal compact

$$U_1(\mathbb{C}) = S_1 \xrightarrow{g} U_n(\mathbb{C})$$
$$e^{i\theta} \longmapsto (g_{ij}(e^{i\theta}))$$



Let $\mathring{G}$ be reductive. The loop group is

$$L\mathring{G} = \mathring{G}(\mathbb{C}[t, t^{-1}])$$

Add the central extension and loop rotation

$$G = \mathbb{C}^\times \ltimes L\mathring{G} \rtimes \mathbb{C}^\times .$$

MIRACLE G is an affine Kac-Moody group
i.e. G is generated by $SL_2(\mathbb{C})$ subgroups

$$\varphi_i : SL_2(\mathbb{C}) \longrightarrow G \quad \text{for } i \in \{0, 1, \dots, n\}.$$

Reductions $\mathfrak{g}$ and $U\mathfrak{g}$

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\exp} & G \\ cX & \longmapsto & e^{cX} \end{array}$$

Lie algebra

$U\mathfrak{g}$ is an associative algebra

$$\begin{tikzcd} G\text{-modules} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{integrable } \mathfrak{g}\text{-modules} \arrow[dl, leftrightarrow] \\ & \text{integrable } U\mathfrak{g}\text{-modules} & \end{tikzcd}$$

Since

$$G = e^{\mathbb{C}K} \ltimes L\mathring{G} \rtimes e^{\mathbb{C}d}$$

is generated by

$$\varphi_0(SL_2(\mathbb{C})), \quad \dots, \quad \varphi_n(SL_2(\mathbb{C}))$$

then $U\mathfrak{g}$ is generated by

$$\begin{array}{l} e_0, \dots, e_n \\ f_0, \dots, f_n \end{array} \quad \text{and } K \text{ and } d.$$

15.10.2020
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Structure of G-modules M

A large commutative subgroup T

$$\begin{matrix} G \\ \cup\!\! \mid \\ T \end{matrix} \quad \text{has Lie algebra} \quad \begin{matrix} \mathfrak{g} = \log(G) \\ \cup\!\! \mid \\ \mathfrak{h} = \log(T) \end{matrix}$$

Symmetries preserving T: The affine Weyl group

$$W = \{\text{automorphisms } w \text{ of } G \text{ such that } w(T) = T\}.$$

Simultaneous eigenvectors for T

$$M = \bigoplus_{\lambda} M_\lambda$$

where

$$\begin{aligned} M_\lambda &= \{m \in M \mid \text{if } e^H \in T \text{ then } e^H m = e^{\lambda(H)} m\} \\ &= \{m \in M \mid \text{if } H \in \mathfrak{h} \text{ then } Hm = \lambda(H) m\} \end{aligned}$$

with

$$\begin{matrix} \lambda : \mathfrak{h} \to \mathbb{C} \\ H \mapsto \lambda(H) \end{matrix} \qquad \text{i.e. } \lambda \in \mathfrak{h}^*.$$

W-symmetry $\quad w : M_\lambda \xrightarrow{\sim} M_{w\lambda}$

Integrality $\quad M_\lambda \neq 0$ only if $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$

where

$$\mathfrak{h}^*_{\mathbb{Z}} = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}\}$$

The example $\mathring{G} = SL_2(\mathbb{C})$

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$G = \mathbb{C}^\times \ltimes \mathring{G}\ltimes\mathbb{C}^\times$. A. Ram

$$T = \left\{ \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix}, e^{cK}, e^{\gamma D} \;\middle|\; \theta, c, \gamma \in \mathbb{C} \right\}$$

$$\mathfrak{h} = \log(T) = \mathrm{span}\{h, K, D\} \qquad \left(\text{with } h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$\mathfrak{h}^* = \mathrm{span}\{\delta, \omega_1, \Lambda_0\}$$

The action of W on $\mathfrak{h}^*$ is generated by

$$s_0 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(these come from

$$\varphi_0 : SL_2(\mathbb{C}) \to G, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto s_0 \qquad \text{and} \qquad \varphi_1 : SL_2(\mathbb{C}) \to G, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto s_1 \Big)$$

Representatives of W-orbits

$$\hat{E} = \hat{E}^+ \cup \hat{E}^0 \cup \hat{E}^-$$

$$\hat{E}^+ = \{ a\delta + \lambda_1\omega_1 + \ell\Lambda_0 \mid \ell \in \mathbb{Z}_{>0},\ \lambda_1 \in \{0, 1, \ldots, \ell\},\ a \in \mathbb{C} \}$$

$$\hat{E}^0 = \{ a\delta + \lambda_1\omega_1 + 0\Lambda_0 \mid \lambda_1 \in \mathbb{Z}_{\geq 0},\ a \in \mathbb{C} \}$$

$$\hat{E}^- = \{ a\delta - \lambda_1\omega_1 - \ell\Lambda_0 \mid \ell \in \mathbb{Z}_{>0},\ \lambda_1 \in \{0, +1, \ldots, \ell\},\ a \in \mathbb{C} \}$$

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Orbits of W on $\mathfrak{h}^*$

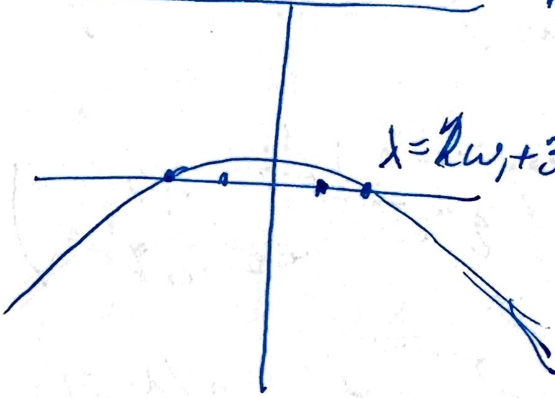
$\mathfrak{h}^* = \mathrm{span}\{\delta, \omega_1, \lambda_0\}$. and

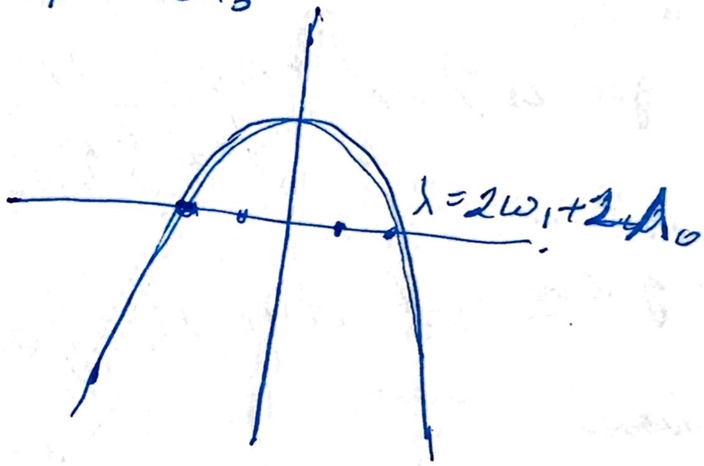
W is generated by the transformations:

$$s_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Calculate orbits

$\lambda = a\delta + \lambda_1\omega_1 + \ell\lambda_0$

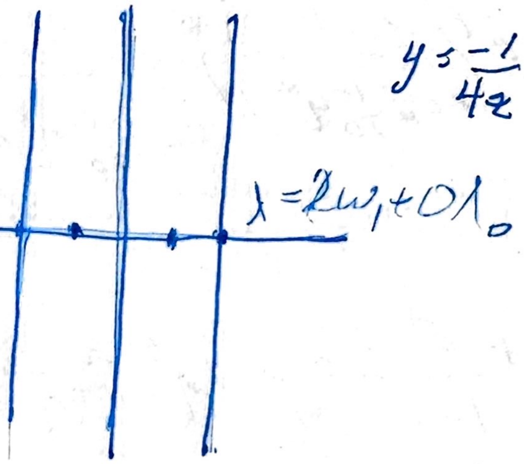




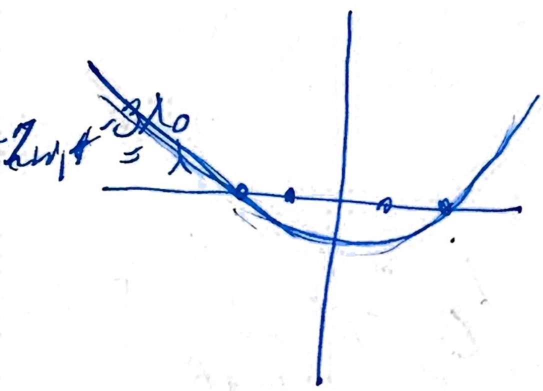
$$y = -\frac{1}{4z}(x-\lambda_1)^2 + a$$

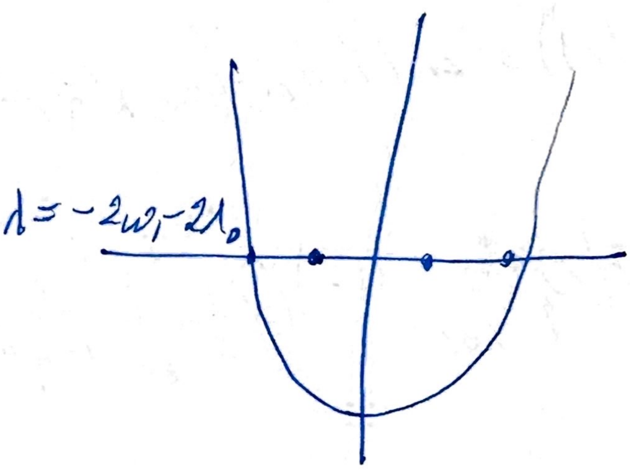
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ t \end{bmatrix}$, $t = -10..10$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ t \end{bmatrix}$, $t = -10..10$



$$y = \frac{-1}{4z}(x-2)^2$$





Formula for the parabola.

Extremal weight modules $L(\lambda)$ for $\lambda \in \mathfrak{E}$.

Start with one vector v_λ.

$$Hv_\lambda = \lambda(H)v_\lambda, \quad \text{for } H \in \mathfrak{h}.$$

Move it around with W $\{v_{w\lambda} \mid w \in W\}$

$$wv_\lambda = v_{w\lambda}.$$

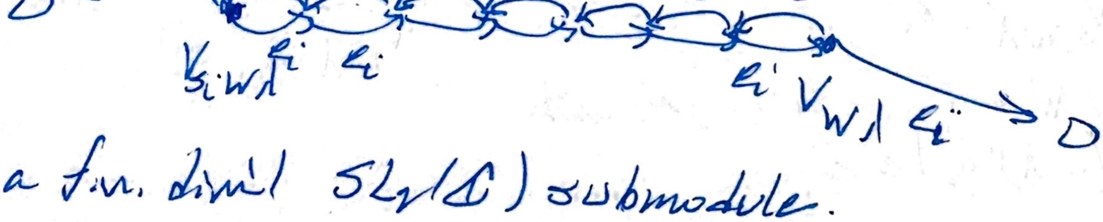
Act with $U\mathfrak{g}$: Linear combinations of products of $e_0, e_1, \ldots, e_n, f_0, f_1, \ldots, f_n$

and put relations

$$0 = f_i v_{s_i w\lambda} = f_i^{(w\lambda)(h_i)+1} v_{w\lambda} \quad \text{and}$$

$$e_i^{(w\lambda)(h_i)+1} v_{s_i w\lambda} = e_i v_{w\lambda} = 0$$

for $w \in W$ and $i \in \{0, 1, \ldots, n\}$ with $(w\lambda)(h_i) \in \mathbb{Z}_{\geq 0}$.



is a fin. dim'l $SL_2(\mathbb{C})$ submodule.

Characters

$$\mathrm{char}(M) = \sum_{\mu} \dim(M_\mu) e^{\mu}$$

a generating function for $\dim(M_\mu)$. Let
$\rho = \omega_1 + \cdots + \omega_n + h^\vee \Lambda_0$ so that $\rho(h_i) = 1$.

Pos. level (Weyl-Kac character)

$$\mathrm{char}(L(\lambda)) = e^{-\rho}\,\mathrm{char}(U^-)\Big(\sum_{w\in W} \det(w)\, e^{w(\lambda+\rho)}\Big)$$

Neg. level (Weyl-Kac character)

$$\mathrm{char}(L(\lambda)) = e^{\rho}\,\mathrm{char}(U^+)\Big(\sum_{w\in W} \det(w)\, e^{w(\lambda-\rho)}\Big)$$

Level 0 (Nakajima-Braverman-Finkelberg-Ion-Fourier-Littelmann)

$$\mathrm{char}(L(\lambda)) = \mathrm{char}(RG_\lambda)\, \tilde{E}_{w_0\lambda}(q^{-1}, 0)$$

where $q^{-1} = e^{\delta}$ and $\tilde{E}_{w_0\lambda}(q^{-1},0)$ is the nonsymm. Macdonald Polynomial and

if $\lambda = m_1\omega_1 + \cdots + m_n\omega_n$ then

$$\mathrm{char}(RG_\lambda) = \Big(\mathcal{O}_{q^{m_1}} \prod_{k=1}^{m_1-1} \frac{1}{1-q^k}\Big)\Big(\cdots\Big(\mathcal{O}_{q^{m_n}} \prod_{k=1}^{m_n-1} \frac{1}{1-q^k}\Big)$$

with

$$\mathcal{O}_q = \frac{q^{-1}}{1-q^{-1}} + \frac{1}{1-q} = \cdots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots$$

Geometry

"Upper triangular" subgroups of $L\mathring{G} = SL_n(\mathbb{C}[\epsilon,\epsilon^{-1}])$.

$$I^+ = \left\{ (g_{ij}) \in SL_n(\mathbb{C}[\epsilon]) \;\middle|\; (g_{ij}(0)) \in \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

$$I^0 = \left\{ (g_{ij}) \in SL_n(\mathbb{C}[\epsilon,\epsilon^{-1}]) \;\middle|\; (g_{ij}) \in \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \right\}.$$

$$I^- = \left\{ (g_{ij}) \in SL_n(\mathbb{C}[\epsilon^{-1}]) \;\middle|\; (g_{ij}(\infty)) \in \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}$$

Affine flag varieties

pos. level G/I^+

level 0 G/I^0

neg. level G/I^-

I^+-orbits

$$G = \bigsqcup_{x\in W} I^+ x I^+$$

$$G = \bigsqcup_{z\in W} I^+ z I^0$$

$$G = \bigsqcup_{y\in W} I^+ y I^-.$$

Borel-Weil-Bott type results

$\lambda \in \mathfrak{h}^*$ determines a line bundle and

$$L(\lambda) \simeq H^0(G/I^+, \mathcal{L}_\lambda), \quad \text{if } \lambda \in \hat{E}^+$$

$$L(\lambda) \simeq H^0(G/I^0, \mathcal{L}_\lambda), \quad \text{if } \lambda \in \hat{E}^0$$

$$L(\lambda) \simeq H^0(G/I^-, \mathcal{L}_\lambda). \quad \text{if } \lambda \in \hat{E}^-$$