Periodic Permutations and Macdonald polynomials

USC Talk ①

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n-periodic permutations

The affine Weyl group W type GL_n

bijections $w: \mathbb{Z} \to \mathbb{Z}$ with $w(i+n) = w(i)+n$.

W is generated by $s_1, \ldots, s_{n-1}$ and π

$\pi(i) = i+1$ for $i \in \mathbb{Z}$,

$s_i(i) = i+1$, $s_i(i+1) = i$ and $s_i(j) = j$ for $j \in \{1, \ldots, n\}$, $j \notin \{i, i+1\}$.

Operators on $\mathbb{Z}^n$

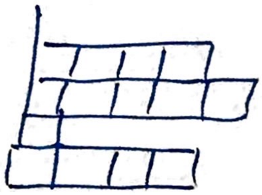
$$s_i(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n)$$

$$\pi(\mu_1, \ldots, \mu_n) = (\mu_n + 1, \mu_1, \ldots, \mu_{n-1})$$

Let u_μ be minimal length

with $u_\mu(0, \ldots, 0) = (\mu_1, \ldots, \mu_n)$.

If $\mu \in \mathbb{Z}^n_{\geq 0}$ draw μ as a configuration of boxes

$\mu = (0, 4, 5, 1, 4) =$  μ_i boxes in row i.

Number box (i,j) with $i+nj$.

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The box greedy reduced word

Find this recursively:

$$u_\mu = \pi\, u_{\pi^{-1}\mu} \quad \text{if } \mu_1 \neq 0,$$

$$u_\mu = s_k\, u_{s_k\mu} \quad \text{if } \mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0 \text{ and } \mu_k \neq 0$$

For example,

$$u_{(0,4,5,1,4)} = (s_1\pi)^6 (s_2 s_1 \pi)(s_3 s_2 s_1 \pi)$$

=

$s_1\pi$	$s_1\pi$	$s_2s_1\pi$	$s_2s_1\pi$	
$s_1\pi$	$s_1\pi$	$s_2s_1\pi$	$s_2s_1\pi$	$s_3s_2s_1\pi$
$s_1\pi$				
$s_1\pi$	$s_2s_1\pi$	$s_2s_1\pi$	$s_2s_1\pi$	

Proposition For a box (i,j) in μ define

$$u_\mu(i,j) = \#\{i' \in \{1,\ldots,i-1\} \mid \mu_{i'} < j \leq \mu_i\} + \#\{i' \in \{i+1,\ldots,n\} \mid \mu_{i'} \leq j-1 < \mu_i\}$$

and

$$b_{i+nj} = s_{u_\mu(i,j)} \cdots s_2 s_1 \pi.$$

Then the box greedy reduced word is

$$u_\mu = \prod_{(i,j)\in\mu} b_{i+nj} \qquad \text{(in increasing order)}.$$

Inversions of u_μ

Let $w \in W$. An inversion of w is $(j,k) \in \mathbb{Z}\times\mathbb{Z}$ with $j<k$ and $w(j) > w(k)$.

Assume $j \in \{1,\ldots,n\}$ and write

$$(j, i+\ell n) = \varepsilon_i^\vee - \varepsilon_j^\vee + \ell K \quad \text{and} \quad \begin{matrix} sh(\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K) = \ell \\ ht(\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K) = j-i \end{matrix}$$

Let

$$Inv(u_\mu) = \left\{ \begin{matrix} \text{inversions } (j,k) \text{ of } u_\mu \\ \text{with } j \in \{1,\ldots,n\} \end{matrix} \right\}$$

Proposition Define

$$v_\mu(i) = \#\{i' \in \{1,\ldots,i-1\} \mid \mu_{i'} \le \mu_i\} + \#\{i' \in \{i+1,\ldots,n\} \mid \mu_{i'} < \mu_i\} + 1$$

and
$$R_\mu(i,j) = \left\{ \begin{matrix} \varepsilon_i^\vee - \varepsilon_{v_\mu(i)} + (\mu_i - j+1)K, \\ \ldots, \varepsilon^\vee_{u_\mu(i,j)} - \varepsilon^\vee_{v_\mu(i)} + (\mu_i - j+1)K \end{matrix} \right\}$$

Then

$$Inv(u_\mu) = \bigcup_{\substack{\text{boxes } (i,j) \\ \text{in } \mu}} R_\mu(i,j)$$

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For example, if $\mu = (10, 4, 5, 1, 4)$ then

$Inv(u_\mu) =$

$\varepsilon_1^\vee - \varepsilon_3^\vee + 4K$	$\varepsilon_1^\vee - \varepsilon_3^\vee + 3K$	$\varepsilon_1^\vee - \varepsilon_3^\vee + 2K$ $\varepsilon_2^\vee - \varepsilon_3^\vee + 2K$	$\varepsilon_1^\vee - \varepsilon_3^\vee + K$ $\varepsilon_2^\vee - \varepsilon_3^\vee + K$	
$\varepsilon_1^\vee - \varepsilon_5^\vee + 5K$	$\varepsilon_1^\vee - \varepsilon_5^\vee + 4K$ $\varepsilon_2^\vee - \varepsilon_5^\vee + 4K$	$\varepsilon_1^\vee - \varepsilon_5^\vee + 3K$ $\varepsilon_2^\vee - \varepsilon_5^\vee + 3K$	$\varepsilon_1^\vee - \varepsilon_5^\vee + 2K$ $\varepsilon_2^\vee - \varepsilon_5^\vee + 2K$	$\varepsilon_1^\vee - \varepsilon_5^\vee + K$ $\varepsilon_2^\vee - \varepsilon_5^\vee + K$ $\varepsilon_3^\vee - \varepsilon_5^\vee + K$
$\varepsilon_1^\vee - \varepsilon_2^\vee + K$				
$\varepsilon_1^\vee - \varepsilon_4^\vee + 4K$	$\varepsilon_1^\vee - \varepsilon_4^\vee + 3K$ $\varepsilon_2^\vee - \varepsilon_4^\vee + 3K$	$\varepsilon_1^\vee - \varepsilon_4^\vee + 2K$ $\varepsilon_2^\vee - \varepsilon_4^\vee + 2K$	$\varepsilon_1^\vee - \varepsilon_4^\vee + K$ $\varepsilon_2^\vee - \varepsilon_4^\vee + K$	

Main point of this talk

Let

$$a(i,j) \text{ and } l(i,j)$$

be the statistics defined by Haglund-Haiman-Loehr. Then

$$\mathrm{ht}\begin{pmatrix}\text{last element}\\ \text{of } R_\mu(i,j)\end{pmatrix} = a(i,j)+1$$

$$\mathrm{sh}\begin{pmatrix}\text{last element}\\ \text{of } R_\mu(i,j)\end{pmatrix} = l(i,j)+1.$$

The HHL formula for ~~the~~ non-symmetric Macdonald polynomials

$$E_\mu(x;q,t) = \sum_{\underbrace{T\in NAF_\mu}_{\text{nonattacking fillings}}} x^T q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} \cdot \prod_{\substack{(i,j)\in\mu \\ T(i,j)\neq T(i,j-1)}} \frac{1-t}{1-q^{\overbrace{l(i,j)}^{\text{leg of box }(i,j)}+1}\, t^{\overbrace{a(i,j)}^{\text{arm of box }(i,j)}+1}}$$

Polynomials

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ define

$$x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}. \quad \text{Then } x^\mu x^\nu = x^{\mu+\nu}$$

and

$$\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \text{ has basis } \{x^\mu \mid \mu \in \mathbb{Z}^n\}$$

$$\mathbb{C}[x_1, \ldots, x_n] \text{ has basis } \{x^\mu \mid \mu \in \mathbb{Z}^n_{\geq 0}\}.$$

Then W acts on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by

$$s_i x^\mu = x^{s_i \mu} \quad \text{and} \quad \pi x^\mu = x^{\pi \mu}.$$

DAHA Operators

Fix $q, t \in \mathbb{C}^\times$ with

$$1 \notin \{q^s t^h \mid (s,h) \in \mathbb{Z}\times\mathbb{Z} - \{(0,0)\}\}.$$

Define operators g and $S_1, \ldots, S_{n-1}$ and σ_π on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by

$$\sigma_\pi = t^{-\frac{1}{2}(n-1)} x_1 S_1 S_2 \cdots S_{n-1}$$

$$S_i = t - \frac{t x_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i) \quad \text{for } i \in \{1, \ldots, n-1\}$$

$$g = s_1 s_2 \cdots s_{n-1} T_{q^{-1}; x_n}$$

where $(T_{q^{-1}; x_n} f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{n-1}, q^{-1} x_n)$.

The <u>Cherednik-Dunkl operators</u> are $y_1, \ldots, y_n$ given by

$$y_i = t^{(i-1) - \frac{1}{2}(n-1)} S_i^{-1} \cdots S_1^{-1} g S_{n-1} S_{n-2} \cdots S_i$$

The <u>intertwiners</u> are $\sigma_1, \ldots, \sigma_{n-1}$ and σ_π with

$$\sigma_i = S_i + \frac{1-t}{1 - y_i^{-1} y_{i+1}} \quad \text{for } i \in \{1, \ldots, n-1\}.$$

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Macdonald polynomials

Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ and

$\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ the decreasing rearrangement of μ

Let $z_\mu \in S_n$ be minimal length with

$\mu = z_\mu \lambda$ and $z_\mu = s_{j_1} \cdots s_{j_k}$ a reduced word.

Let $u_\mu = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for u_μ

The <u>nonsymmetric Macdonald polynomial</u> E_μ is

$$E_\mu = t^{-\frac{1}{2}\sum_{i<j}(\lambda_i - \lambda_j)} \sigma_{i_1} \cdots \sigma_{i_\ell} x^0$$

The <u>permuted basement Macdonald poly.</u> f_μ is

$$f_\mu = s_{j_1} \cdots s_{j_k} E_\lambda$$

The <u>symmetric Macdonald polynomial</u> P_λ is

$$P_\lambda = \sum_{\substack{\nu \text{ rearrangements} \\ \text{of } \lambda}} f_\nu$$