Examples in affine Combinatorial Representation Theory

Talk 1: Examples of Macdonald polynomials

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The affine Weyl group W acts on $\mathbb{Z}^n$

Generators: $s_1, s_2, \ldots, s_{n-1}, \pi$

$\pi(\mu_1, \ldots, \mu_n) = (\mu_n + 1, \mu_1, \ldots, \mu_{n-1})$

$s_i(\mu_1, \ldots, \mu_{i-1}, \mu_i, \mu_{i+1}, \mu_{i+2}, \ldots, \mu_n)$
$= (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n).$

$s_1, s_2, \ldots, s_{n-1}$ generate S_n.

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$.

- $v_\mu \in S_n$ is minimal length such that $v_\mu \mu$ is increasing
- $u_\mu \in W$ is minimal length such that

$$u_\mu(0, \ldots, 0) = (\mu_1, \ldots, \mu_n)$$

<u>Example</u> $\mu = (0,1,2)$ has

$v_\mu = id$.

$$(0,1,2) \xrightarrow{s_1} (1,0,2) \xrightarrow{\pi^{-1}} (0,2,0)$$

$$\xrightarrow{s_1} (2,0,0) \xrightarrow{\pi^{-1}} (0,0,1)$$

$$\xrightarrow{s_2} (0,1,0) \xrightarrow{s_1} (1,0,0) \xrightarrow{\pi^{-1}} (0,0,0).$$

So

$$u_\mu = s_1\pi\, s_1\pi\, s_2 s_1 \pi$$

$$= \begin{array}{c|c|c|} 1 & \multicolumn{2}{l}{} \\ \cline{2-2} 2 & s_1\pi & \multicolumn{1}{c}{} \\ \cline{2-3} 3 & s_1\pi & s_2 s_1 \pi \\ \cline{2-3} \end{array}$$

μ_i boxes in row i

The DAHA $\hat{H}$

Fix $t^{\frac{1}{2}}, q \in \mathbb{C}^\times$.

Generators: $q^\vee, T_1, T_2, \ldots, T_{n-1}, g$

Relations:

Define $X_1, \ldots, X_n$ and

$Y_1, \ldots, Y_n$ by

$Y_1 = g T_{n-1} \cdots T_1$ and $Y_{j+1} = T_j^{-1} Y_j T_j^{-1}$,

$X_1 = q^\vee T_{n-1}^{-1} \cdots T_1^{-1}$ and $X_{j+1} = T_j X_j T_j$.

Proposition

$$X_i X_j = X_j X_i \quad \text{and} \quad Y_i Y_j = Y_j Y_i$$

Intertwiners

$\tau_\pi^\vee = q^\vee$ and

$$t^{\frac{1}{2}}\tau_i^\vee = t^{\frac{1}{2}}T_i + \frac{(1-t)}{1-y_i^{-1}y_{i+1}}$$

Proposition Let $y_{i+n} = q y_i$.
Then

$$y_i \tau_\pi^\vee = \tau_\pi^\vee y_{i-1} \quad \text{for } i \in \mathbb{Z},$$

$$y_i \tau_i^\vee = \tau_i^\vee y_{i+1}$$

and

$$y_{i+1} \tau_i^\vee = \tau_i^\vee y_i$$

$$y_j \tau_i^\vee = \tau_i^\vee y_j \quad \text{for } j \in \{1, \ldots, n\}$$
with $j \notin \{i, i+1\}$.

Macdonald polynomials E_μ

The polynomial representation

$$\mathbb{C}[X] = \mathrm{Ind}_H^{\tilde{H}}(\mathbb{1}) \quad \text{with}$$

$$T_i \mathbb{1} = t^{\frac{1}{2}} \mathbb{1} \quad \text{and} \quad q\mathbb{1} = \mathbb{1}.$$

Let $X^\mu = X_1^{\mu_1} \cdots X_n^{\mu_n}$ and

$$x^\mu = X^\mu \mathbb{1}.$$

Then $\mathbb{C}[X]$ has basis

$$\{x^\mu \mid \mu \in \mathbb{Z}^n\}.$$

The nonsymmetric Macdonald polynomial E_μ is

$$E_\mu = t^{-\frac{1}{2}\ell(v_\mu)} \tau^\vee_{u_\mu} \mathbb{1}$$

where

$$\tau^\vee_{u_\mu} = \tau^\vee_{i_1} \cdots \tau^\vee_{i_\ell} \quad \text{if}$$

$u_\mu = s_{i_1} \cdots s_{i_\ell}$ is a reduced word.

Theorem

(a) The E_μ are simultaneous eigenvectors for $Y_1, \ldots, Y_n$,

$$Y_i \cdot E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1) + \frac{1}{2}(n-1)} E_\mu$$

(b) The coefficient of x^μ in E_μ is 1,

$$E_\mu = x^\mu + \text{lower terms.}$$

The <u>q-Iwahori-Whittaker function</u> is

$E_\mu(q,t)$ specialised at $t=0$,

i.e., $\quad E_\mu(q,0)$

The relations we really need

Lemma Let $w = (w(1), \dots, w(n)) \in S_n$ and

$$\ell(w) = \#\{(i,j) \mid i<j \text{ and } w(i) > w(j)\}$$

Then

(a) $$\left(t^{\frac{1}{2}\ell(w)} T_w\right)\left(t^{\frac{1}{2}} T_i\right) = t^{\frac{1}{2}\ell(ws_i)} T_{ws_i} \quad \text{when}$$

$w = (\cdots i \cdots i+1 \cdots n)$ and $ws_i = (\cdots i+1 \cdots i \cdots n)$

(b) $$\left(t^{\frac{1}{2}\ell(w)} T_w\right)\left(t^{\frac{1}{2}} T_i^{-1}\right) = t\left(t^{\frac{1}{2}\ell(ws_i)} T_{ws_i}\right) \quad \text{when}$$

$w = (\cdots i+1 \cdots i \cdots n)$ and $ws_i = (\cdots i \cdots i+1 \cdots n)$

(c) $$\left(t^{\frac{1}{2}\ell(w)} T_w\right) \tau_\pi^\vee = t^{(w^{-1}(1)-1) - \frac{1}{2}(n-1)} x_{w^{-1}(1)} \left(t^{\ell(v)} T_v\right)$$

with

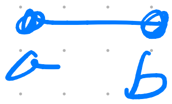
$v = (w(i)-1, \dots, w(n)-1)$

(the entries of v are mod n).

Also

$$t^{\frac{1}{2}} \tau_i^\vee = t^{\frac{1}{2}} T_i + \frac{1-t}{1-y_i^{-1}y_{i+1}} = t^{\frac{1}{2}} T_i^{-1} + \frac{(1-t)y_i^{-1}y_{i+1}}{1-y_i^{-1}y_{i+1}}$$

The affine Weyl group W of type GL_n



means $aba = bab$

means $ab = ba$

Generators: $s_1, s_2, \ldots, s_{n-1}, \pi$

Relations:

$$\pi s_0 \pi^{-1} = s_1$$
$$\pi s_i \pi^{-1} = s_{i+1}$$
$$\pi s_{n-1} \pi^{-1} = s_0$$

$$s_i^2 = 1 \text{ for } i \in \{1, \ldots, n-1\}$$

Another presentation of W

Let $s_1, \ldots, s_{n-1}$ act on $\mathbb{Z}^n$ by

$$s_i(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n)$$

$$W = \{t_\mu w \mid \mu \in \mathbb{Z}^n \text{ and } w \in S_n\}$$

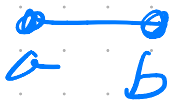
with

$$(t_\mu v)(t_\nu w) = t_{\mu + v\nu}(vw)$$

so that

$$t_\mu t_\nu = t_{\mu+\nu} \text{ and } v t_\nu = t_{v\nu} v$$

Double affine Hecke algebra $\tilde{H}$



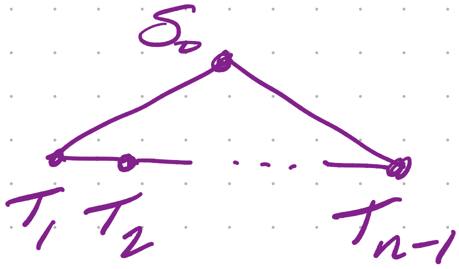
means $aba = bab$

means $ab = ba$

Generators: $g^\vee, T_1, \dots, T_{n-1}, g$

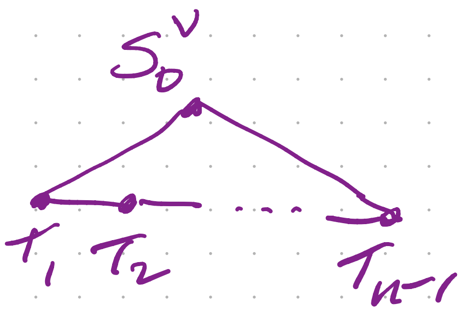
and $q, t^{\frac{1}{2}} \in \mathbb{C}^\times$

Relations:



$$g S_0 g^{-1} = T_1$$

$$g T_i g^{-1} = T_{i+1}, \quad g T_{n-1} g^{-1} = S_0$$



$$g^\vee S_0^\vee (g_0^\vee)^{-1} = T_1$$

$$g^\vee T_i (g^\vee)^{-1} = T_{i+1},$$

$$g^\vee T_{n-1} (g^\vee)^{-1} = S_0^\vee$$

$$T_1 g^\vee g = g g^\vee T_{n-1}^{-1}$$

$$T_{n-1}^{-1} \cdots T_1^{-1} g (g^\vee)^{-1} = q (g^\vee)^{-1} g T_{n-1} \cdots T_1$$

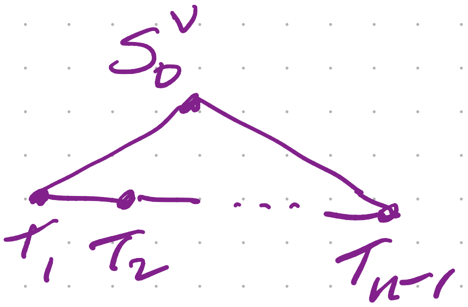
$$(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0$$

Another presentation of $\hat{H}$

Keep $q, t^{\frac{1}{2}} \in \mathbb{C}^\times$.

Generators: $g^\vee, T_1, \dots, T_{n-1}, Y_1, \dots, Y_n$

Relations:



$$g^\vee S_0^\vee (g_0^\vee)^{-1} = T_1$$

$$g^\vee T_i (g^\vee)^{-1} = T_{i+1},$$

$$g^\vee T_{n-1} (g^\vee)^{-1} = S_0^\vee$$

$Y_{i+1} = T_i^{-1} Y_i T_i^{-1}$, for $i \in \{1, \dots, n-1\}$

$T_i Y_j = Y_j T_i$, and $j \notin \{i, i+1\}$

$Y_i Y_j = Y_j Y_i$ for $i, j \in \{1, \dots, n\}$

$Y_i g^\vee = g^\vee Y_{i-1}$ and $Y_1 g^\vee = g^\vee q^{-1} Y_n$
for $i \in \{2, \dots, n\}$.

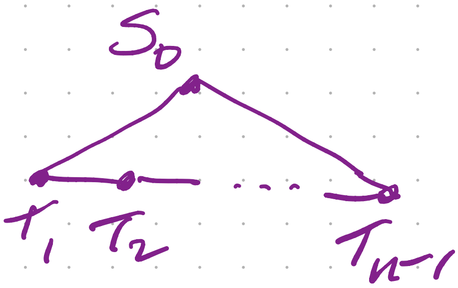
$$(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0$$

And ANOTHER presentation of $\hat{H}$

Keep $q, t^{1/2} \in \mathbb{C}^\times$

Generators: $X_1, X_2, \dots X_n, T_1, \dots, T_{n-1}, g$

Relations:



$$g S_0 g^{-1} = T_1$$

$$g T_i g^{-1} = T_{i+1},$$

$$g T_{n-1} g^{-1} = S_0$$

$X_{i+1} = T_i X_i T_i$, for $i \in \{1, \dots, n-1\}$

$T_i X_j = X_j T_i$, and $j \notin \{i, i+1\}$

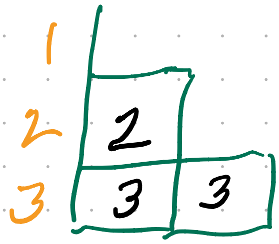
$X_i X_j = X_j X_i$ for $i, j \in \{1, \dots, n\}$

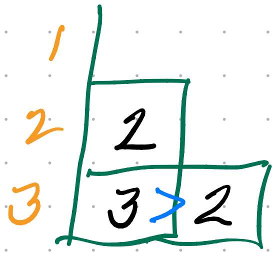
$X_i g = g X_{i-1}$ and $X_1 g = g q^{+1} X_n$ for $i \in \{2, \dots, n\}$.

$$(T_i - t^{1/2})(T_i + t^{-1/2}) = 0$$

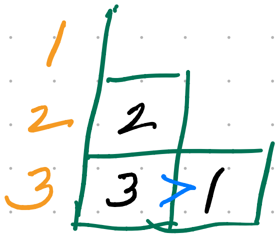
The final answer for $E_{(0,1,2)}$

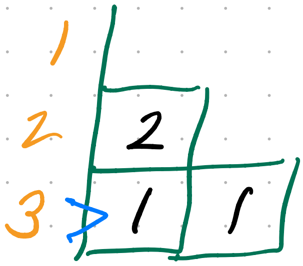
$$E_{(0,1,2)} = x_2 x_3^2 + \left(\frac{1-t}{1-qt}\right) x_2^2 x_3$$



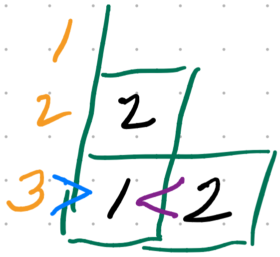


$$+ \left(\frac{1-t}{1-qt}\right) x_1 x_2 x_3 + \left(\frac{1-t}{1-q^2t^2}\right) t\, x_1^2 x_2$$

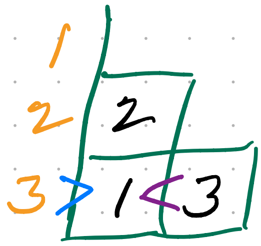




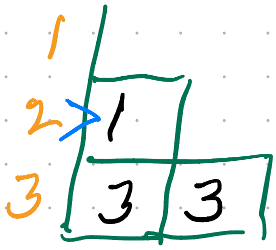
$$+ \left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right) qt\, x_1 x_2^2$$

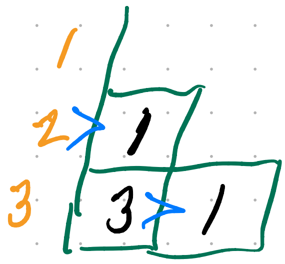


$$+ \left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right) qt\, x_1 x_2 x_3$$

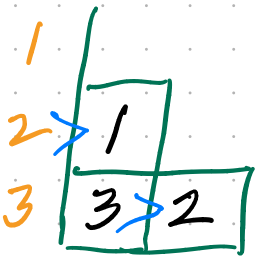


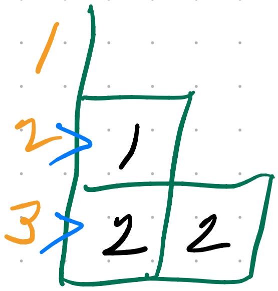
$$+\left(\frac{1-t}{1-qt}\right) x_1 x_3^2 + \left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right) x_1^2 x_3$$



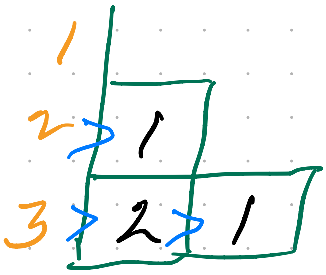


$$+\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt}\right) x_1 x_2 x_3 + \left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right) x_1 x_2^2$$

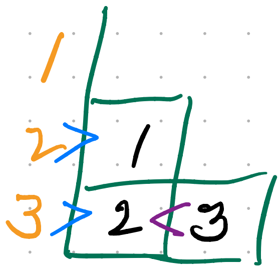




$$+\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt}\right) x_1^2 x_2$$



$$+\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt}\right) q\, x_1 x_2 x_3$$



Setting $t=0$ in $E_{(0,1,2)}$

$$E_{(0,1,2)} = x_2 x_3^2 + \left(\frac{1-\cancel{t}}{1-q\cancel{t}}\right) x_2^2 x_3$$

$$\begin{array}{c|cc} 1 & & \\ 2 & 2 & \\ 3 & 3 & 3 \end{array} \qquad\qquad \begin{array}{c|cc} 1 & & \\ 2 & 2 & \\ 3 & 3 & 2 \end{array}$$

$$+ \left(\frac{1-\cancel{t}}{1-q\cancel{t}}\right) x_1 x_2 x_3 + \cancel{\left(\frac{1-t}{1-q^2t^2}\right) t\, x_1^2 x_2}$$

$$\begin{array}{c|cc} 1 & & \\ 2 & 2 & \\ 3 & 3 & 1 \end{array} \qquad\qquad \begin{array}{c|cc} 1 & & \\ 2 & 2 & \\ 3 & 1 & 1 \end{array}$$

$$+ \cancel{\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right) qt\, x_1 x_2^2} \qquad \begin{array}{c|cc} 1 & & \\ 2 & 2 & \\ 3 & 1 & 2 \end{array}$$

$$+ \cancel{\left(\frac{1-t}{1-q^2t^2}\right)\left(\frac{1-t}{1-qt}\right) qt\, x_1 x_2 x_3} \qquad \begin{array}{c|cc} 1 & & \\ 2 & 2 & \\ 3 & 1 & 3 \end{array}$$

$$+\left(\frac{1-\cancel{t}}{1-\cancel{qt}}\right)x_1x_3^2+\left(\frac{1-\cancel{t}}{1-\cancel{q^2t^2}}\right)\left(\frac{1-\cancel{t}}{1-\cancel{qt}}\right)x_1^2x_3$$

$$\begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 3 & 3 \end{array} \qquad\qquad \begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 3 & 1 \end{array}$$

$$+\left(\frac{1-\cancel{t}}{1-\cancel{qt}}\right)\left(\frac{1-\cancel{t}}{1-\cancel{qt}}\right)x_1x_2x_3+\left(\frac{1-\cancel{t}}{1-\cancel{q^2t^2}}\right)\left(\frac{1-\cancel{t}}{1-\cancel{qt}}\right)x_1x_2^2$$

$$\begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 3 & 2 \end{array} \qquad\qquad \begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 2 & 2 \end{array}$$

$$+\left(\frac{1-\cancel{t}}{1-\cancel{q^2t^2}}\right)\left(\frac{1-\cancel{t}}{1-\cancel{qt}}\right)\left(\frac{1-\cancel{t}}{1-\cancel{qt}}\right)x_1^2x_2 \qquad \begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 2 & 1 \end{array}$$

$$+\left(\frac{1-\cancel{t}}{1-\cancel{q^2t^2}}\right)\left(\frac{1-\cancel{t}}{1-\cancel{qt}}\right)\left(\frac{1-\cancel{t}}{1-\cancel{qt}}\right)q\,x_1x_2x_3 \qquad \begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 2 & 3 \end{array}$$

The final answer for $E_{(0,1,2)}(q,0)$

$$E_{(0,1,2)}(q,0) = x_2x_3^2 + x_2^2x_3 + x_1x_3^2$$

$$\begin{array}{c|cc} 1 & & \\ 2 & 2 & \\ 3 & 3 & 3 \end{array} \qquad \begin{array}{c|cc} 1 & & \\ 2 & 2 & \\ 3 & 3 & 2 \end{array} \qquad \begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 3 & 3 \end{array}$$

$$+ x_1^2x_3 + x_1x_2^2 + x_1^2x_2$$

$$\begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 3 & 1 \end{array} \qquad \begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 2 & 2 \end{array} \qquad \begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 2 & 1 \end{array}$$

$$+ x_1x_2x_3 + q\,x_1x_2x_3 + x_1x_2x_3$$

$$\begin{array}{c|cc} 1 & & \\ 2 & 2 & \\ 3 & 3 & 1 \end{array} \qquad \begin{array}{c|cc} 1 & & \\ 2 & 2 & \\ 3 & 1 & 3 \end{array} \qquad \begin{array}{c|cc} 1 & & \\ 2 & 1 & \\ 3 & 3 & 2 \end{array}$$

The module $L(\omega_1+\omega_2)$ for $U_t(\hat{\mathfrak{sl}}_3)$

Basis:

$\epsilon_1^{r_1}\epsilon_2^{r_2}v_{\alpha_1}$, $\epsilon_1^{r_1}\epsilon_2^{r_2}v_{-\alpha_2}$,

$\epsilon_1^{r_1}\epsilon_2^{r_2}v_{\theta}$,

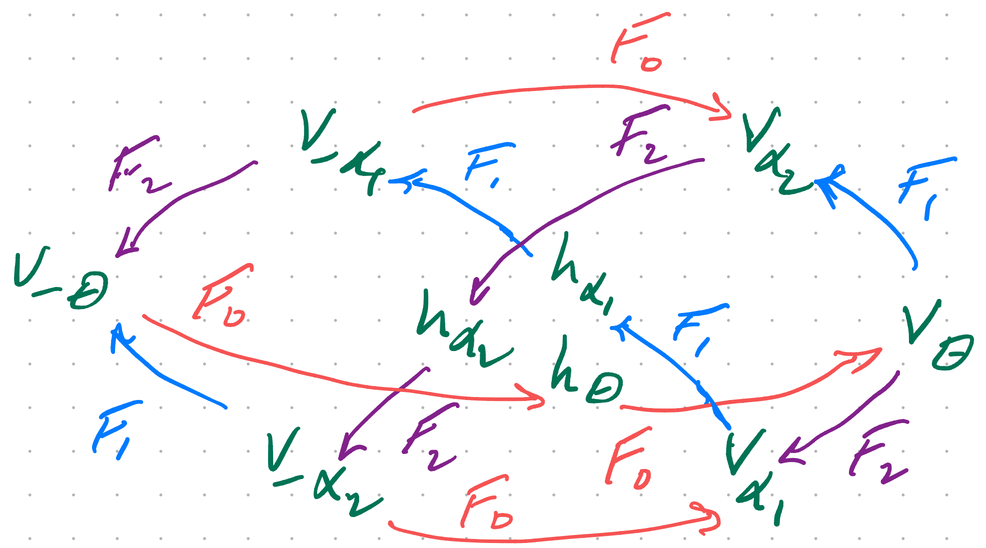
$\epsilon_1^{r_1}\epsilon_2^{r_2}v_{-\theta}$,

$\epsilon_1^{r_1}\epsilon_2^{r_2}v_{\alpha_2}$, $\epsilon_1^{r_1}\epsilon_2^{r}v_{-\alpha_1}$,

$\epsilon_1^{r_1}\epsilon_2^{r_2}h_{\theta}$, $\epsilon_1^{r_1}\epsilon_2^{r_2}h_{\alpha_1}$, $\epsilon_1^{r_1}\epsilon_2^{r_2}h_{\alpha_2}$

with $r_1, r_2 \in \mathbb{Z}$.

Kac-Moody action



after setting $\epsilon_1=1$ and $\epsilon_2=1$.