Examples in affine Combinatorial Representation Theory

Talk 2: Extremal weight modules in Level 0

Arun Ram
University of Melbourne

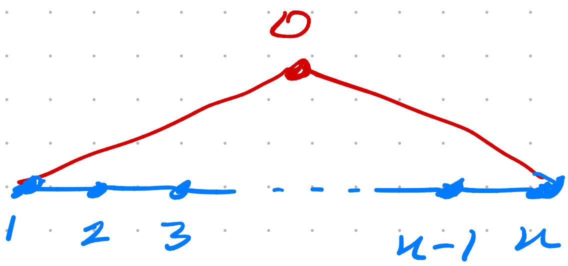
IISc Bengaluru
DMRT2020

11 December 2020

Positive level, negative level and level zero
arXiv1907.11796
with Finn McGlade and Yaping Yang

Dynkin diagrams

finite Dynkin diagram $\mathring{\mathfrak{g}} = \mathfrak{sl}_{n+1}$



affine Dynkin diagram $\mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}$

$$\mathfrak{g} = \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

Fundamental Weights

$\omega_1, \dots, \omega_n$ fundamental weights for $\mathring{\mathfrak{g}}$

$\Lambda_0, \Lambda_1, \dots, \Lambda_n$ fundamental weights for $\mathfrak{g}$

For $i \in \{1, \dots, n\}$,

$$\Lambda_i = \omega_i + \Lambda_0$$

Indexing extremal weight modules

Define

$$(\mathfrak{h}_{\mathbb{Z}}^*)^{pos} = \mathbb{Z}_{\geq 0}\text{-span}\{\Lambda_0, \Lambda_1, \ldots, \Lambda_n\}$$

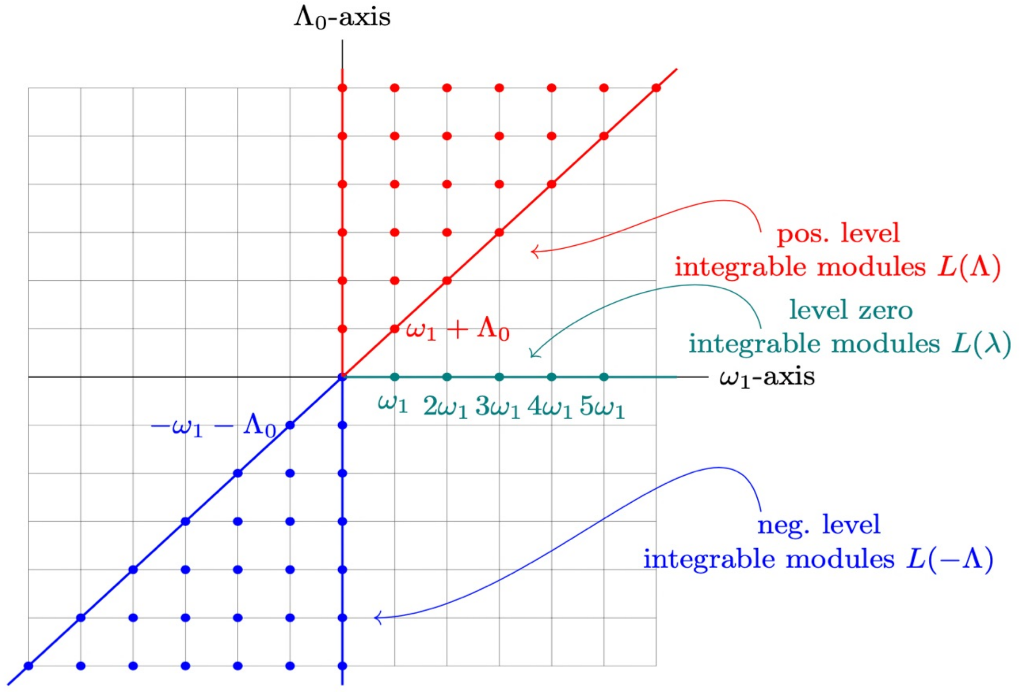
$$(\mathfrak{h}_{\mathbb{Z}}^*)^{0} = \mathbb{Z}_{\geq 0}\text{-span}\{\omega_1, \ldots, \omega_n\}$$

$$(\mathfrak{h}_{\mathbb{Z}}^*)^{neg} = \mathbb{Z}_{\leq 0}\text{-span}\{\Lambda_0, \Lambda_1, \ldots, \Lambda_n\}.$$

Then

$$\left\{\begin{matrix}\text{extremal weight} \\ \text{modules for } U_t\mathfrak{g}\end{matrix}\right\} \longleftrightarrow (\mathfrak{h}_{\mathbb{Z}}^*)^{pos} \cup (\mathfrak{h}_{\mathbb{Z}}^*)^{0} \cup (\mathfrak{h}_{\mathbb{Z}}^*)^{neg}$$

$$L(\lambda) \longleftarrow\!\!| \ \lambda$$



Generators and relations for $U_t\mathfrak{g}$

Kac-Moody presentation

Generators: $C^{\pm\frac{1}{2}}, K_0^{\pm 1}, \ldots, K_n^{\pm 1}, D^{\pm 1}$

$$E_0, E_1, \ldots, E_n$$

$$F_0, F_1, \ldots, F_n$$

Relations:

Drinfeld:

"Fortunately $U_t\mathfrak{g}$ has another presentation..."

Kac-Moody presentation of $U_t\mathfrak{g}$

Relations:

$$C^{\frac{1}{2}} D = D C^{\frac{1}{2}}, \quad C^{\frac{1}{2}} K_i = K_i C^{\frac{1}{2}},$$

$$D K_i = K_i D, \quad K_i K_j = K_j K_i,$$

$$C^{\frac{1}{2}} E_i C^{-\frac{1}{2}} = E_i, \quad C^{\frac{1}{2}} F_i C^{-\frac{1}{2}} = F_i,$$

$$D E_0 D^{-1} = t E_0, \quad D E_i D^{-1} = E_i, \text{ for}$$

$$D F_0 D^{-1} = t^{-1} F_0, \quad D F_i D^{-1} = F_i, \quad i \in \{1, \dots, n\}$$

$$K_i E_j K_i^{-1} = t^{c_{ij}} E_j, \quad K_i F_j K_i^{-1} = t^{-c_{ij}} F_j$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{t - t^{-1}}$$

$$E_i^2 E_{i\pm1} - (t + t^{-1}) E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0,$$

$$F_i^2 F_{i+1} - (t + t^{-1}) F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0$$

where c_{ij} are the entries of

$$\begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ -1 & & & & -1 & 2 \end{pmatrix}$$

(other entries are 0)

Loop presentation

Generators: $C^{\pm\frac{1}{2}}, K_1^{\pm 1}, \dots, K_n^{\pm 1}, D^{\pm 1}$

$x_{1,r}^{+}, \dots, x_{n,r}^{+}$ with $r \in \mathbb{Z}$

$x_{1,r}^{-}, \dots, x_{n,r}^{-}$ with $r \in \mathbb{Z}$

$e_s^{(1)}, \dots, e_s^{(n)}$ with $s \in \mathbb{Z}_{\neq 0}$.

Relations:

$$\exp\left(\sum_{\ell \in \mathbb{Z}_{>0}} \frac{P_\ell^{(i)}}{[\ell]} z^\ell\right) = 1 + \sum_{s \in \mathbb{Z}_{>0}} e_s^{(i)} z^s$$

$$\exp\left(\sum_{\ell \in \mathbb{Z}_{>0}} \frac{P_{-\ell}^{(i)}}{[\ell]} z^{-\ell}\right) = 1 + \sum_{s \in \mathbb{Z}_{>0}} e_{-s}^{(i)} z^{-s}$$

$$[P_s^{(i)}, P_r^{(j)}] = \delta_{r,-s} \frac{[C_{ij} s]}{s} \frac{C^s - C^{-s}}{t - t^{-1}}$$

Loop presentation of $U_t\mathfrak{g}$

Relations:

$$C^{\frac{1}{2}}D = DC^{\frac{1}{2}}, \quad C^{\frac{1}{2}}K_i = K_i C^{\frac{1}{2}},$$

$$DK_i = K_i D, \quad K_i K_j = K_j K_i,$$

$$K_i p_s^{(j)} = p_s^{(j)} K_i, \quad K_i x_{j,r}^+ K_i^{-1} = t^{c_{ij}} x_{j,r}^+$$

$$K_i x_{j,r}^- K_i^{-1} = t^{-c_{ij}} x_{j,r}^-$$

$$[p_s^{(i)}, x_{j,r}^+] = \frac{[c_{ij} s]}{s} C^{-\frac{1}{2}|s|} x_{j,r+s}^+$$

$$[p_s^{(i)}, x_{j,r}^-] = \frac{-[c_{ij} s]}{s} C^{\frac{1}{2}|s|} x_{j,r+s}^-$$

$$[p_s^{(i)}, p_r^{(j)}] = \delta_{r,-s} \frac{[c_{ij} s]}{s} \frac{C^s - C^{-s}}{t - t^{-1}}$$

$$[x_{i,r}^+, x_{j,s}^-] = \delta_{ij} \frac{C^{\frac{1}{2}(r-s)} q_{r+s}^{(i)} - C^{-\frac{1}{2}(r-s)} q_{r+s}^{(i)}}{t - t^{-1}}$$

and

$$x^+_{i,r+1} x^+_{j,s} - t^{c_{ij}} x^+_{j,s} x^+_{i,r+1} = t^{c_{ij}} x^+_{i,r} x^+_{j,s+1} - x^+_{j,s+1} x^+_{i,r},$$

$$x^-_{i,r+1} x^-_{j,s} - t^{c_{ij}} x^-_{j,s} x^-_{i,r+1} = t^{c_{ij}} x^-_{i,r} x^-_{j,s+1} - x^-_{j,s+1} x^-_{i,r},$$

$$x^+_{i,r_1} x^+_{i,r_2} x^+_{j,s} - (t+t^{-1}) x^+_{i,r_1} x^+_{j,s} x^+_{i,r_2} + x^+_{j,s} x^+_{i,r_1} x^+_{i,r_2} + x^+_{i,r_2} x^+_{i,r_1} x^+_{j,s} - (t-t^{-1}) x^+_{i,r_2} x^+_{j,s} x^+_{i,r_1} + x^+_{j,s} x^+_{i,r_2} x^+_{i,r_1} = 0,$$

$$x^-_{i,r_1} x^-_{i,r_2} x^-_{j,s} - (t+t^{-1}) x^-_{i,r_1} x^-_{j,s} x^-_{i,r_2} + x^-_{j,s} x^-_{i,r_1} x^-_{i,r_2} + x^-_{i,r_2} x^-_{i,r_1} x^-_{j,s} - (t-t^{-1}) x^-_{i,r_2} x^-_{j,s} x^-_{i,r_1} + x^-_{j,s} x^-_{i,r_2} x^-_{i,r_1} = 0$$

with $P^{(ij)}_s$ defined by

Let

$$[r] = \frac{t^r - t^{-r}}{t - t^{-1}}$$ and define

$p_\ell^{(i)}$ and $q_\ell^{(i)}$ for $\ell \in \mathbb{Z}_{\neq 0}$ by

$$\exp\left(\sum_{\ell \in \mathbb{Z}_{>0}} \frac{p_\ell^{(i)}}{[\ell]} z^\ell\right) = 1 + \sum_{s \in \mathbb{Z}_{>0}} e_s^{(i)} z^s$$

$$\exp\left(\sum_{\ell \in \mathbb{Z}_{>0}} \frac{p_{-\ell}^{(i)}}{[\ell]} z^{-\ell}\right) = 1 + \sum_{s \in \mathbb{Z}_{>0}} e_{-s}^{(i)} z^{-s}$$

$$\sum_{\ell \in \mathbb{Z}_{\geq 0}} q_\ell^{(i)} z^\ell = K_i \exp\left((t - t^{-1}) \sum_{s \in \mathbb{Z}_{>0}} p_s^{(i)} z^s\right)$$

$$\sum_{\ell \in \mathbb{Z}_{\geq 0}} q_{-\ell}^{(i)} z^{-\ell} = K_i^{-1} \exp\left(-(t - t^{-1}) \sum_{s \in \mathbb{Z}_{>0}} p_{-s}^{(i)} z^{-s}\right)$$

Kac-Moody presentation for $U_t\mathfrak{g}$

Generators: $C^{\pm\frac{1}{2}}$ and $D^{\pm 1}$ and

$$K_0^{\pm 1}, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$$

$$E_0, E_1, \ldots, E_n$$

$$F_0, F_1, \ldots, F_n$$

Drinfeld:

"Fortunately $U_t\mathfrak{g}$ has another presentation..."

Loop presentation for $U_t\mathfrak{g}$

Generators: $C^{\pm\frac{1}{2}}$ and $D^{\pm 1}$ and

$$K_1^{\pm 1}, \ldots, K_n^{\pm 1}$$

$$x_{1,r}^{+}, \ldots, x_{n,r}^{+} \qquad \text{for } r \in \mathbb{Z}$$

$$x_{1,r}^{-}, \ldots, x_{n,r}^{-}$$

$$e_s^{(1)}, \ldots, e_s^{(n)} \qquad \text{with } s \in \mathbb{Z}_{\neq 0}.$$

Presentation of $L(\lambda)$ for $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^{\circ}$

$$\lambda = m_1\omega_1 + \cdots + m_n\omega_n$$

Kac-Moody presentation (Kashiwara)

Generators: $\{u_{w\lambda} \mid w \in W\}$

Relations:

$$c^{\frac{1}{2}}u_{w\lambda} = u_{w\lambda}, \qquad K_i u_{w\lambda} = t^{\langle w\lambda, \alpha_i^\vee\rangle} u_{w\lambda}$$

If $\langle w\lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$ then

$$E_i u_{w\lambda} = 0 \text{ and } F_i^{\langle w\lambda, \alpha_i^\vee\rangle} u_{w\lambda} = u_{s_i w\lambda}$$

If $\langle w\lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\leq 0}$ then

$$F_i u_{w\lambda} = 0 \text{ and } E_i^{-\langle w\lambda, \alpha_i^\vee\rangle} u_{w\lambda} = u_{s_i w\lambda}$$

Loop presentation (Drinfeld-Frenkel-Reshetikhin-Chari-Pressley-Nakajima)

Generator: u_λ

Relations: $c^{\frac{1}{2}}u_\lambda = u_\lambda, \quad K_i u_\lambda = t^{m_i} u_\lambda$

$$x_{i,r}^{+} u_\lambda = 0 \text{ for } i \in \{1, \ldots, n\}, \ r \in \mathbb{Z}$$

$$e_s^{(i)} u_\lambda = 0 \text{ for } i \in \{1, \ldots, n\}, \ s \in \mathbb{Z}_{>m_i}$$

$$e_{-s}^{(i)} u_\lambda = 0 \text{ for } i \in \{1, \ldots, n\}, \ s \in \mathbb{Z}_{>m_i}$$

The module $L(\omega_1+\omega_2)$ for $U_t(\hat{\mathfrak{sl}}_3)$

Basis: $\epsilon_1^{r_1}\epsilon_2^{r_2} v_{\alpha_1}$, $\epsilon_1^{r_1}\epsilon_2^{r_2} v_{-\alpha_2}$,

$\epsilon_1^{r_1}\epsilon_2^{r_2} h_{\alpha_1}$, $\epsilon_1^{r_1}\epsilon_2^{r_2} h_{\alpha_2}$

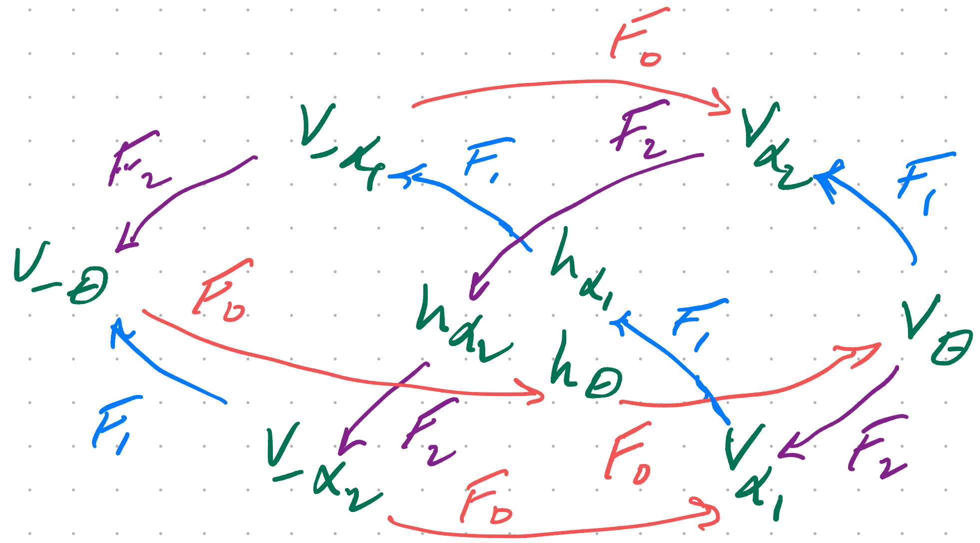
$\epsilon_1^{r_1}\epsilon_2^{r_2} v_{-\theta}$, $\epsilon_1^{r_1}\epsilon_2^{r_2} v_{\theta}$,

$\epsilon_1^{r_1}\epsilon_2^{r_2} h_{\theta}$,

$\epsilon_1^{r_1}\epsilon_2^{r_2} v_{\alpha_2}$, $\epsilon_1^{r_1}\epsilon_2^{r} v_{-\alpha_1}$,

with $r_1, r_2 \in \mathbb{Z}$.

Kac-Moody action



after setting $\epsilon_1 = 1$ and $\epsilon_2 = 1$.

The global Weyl module is the same as the extremal weight module.

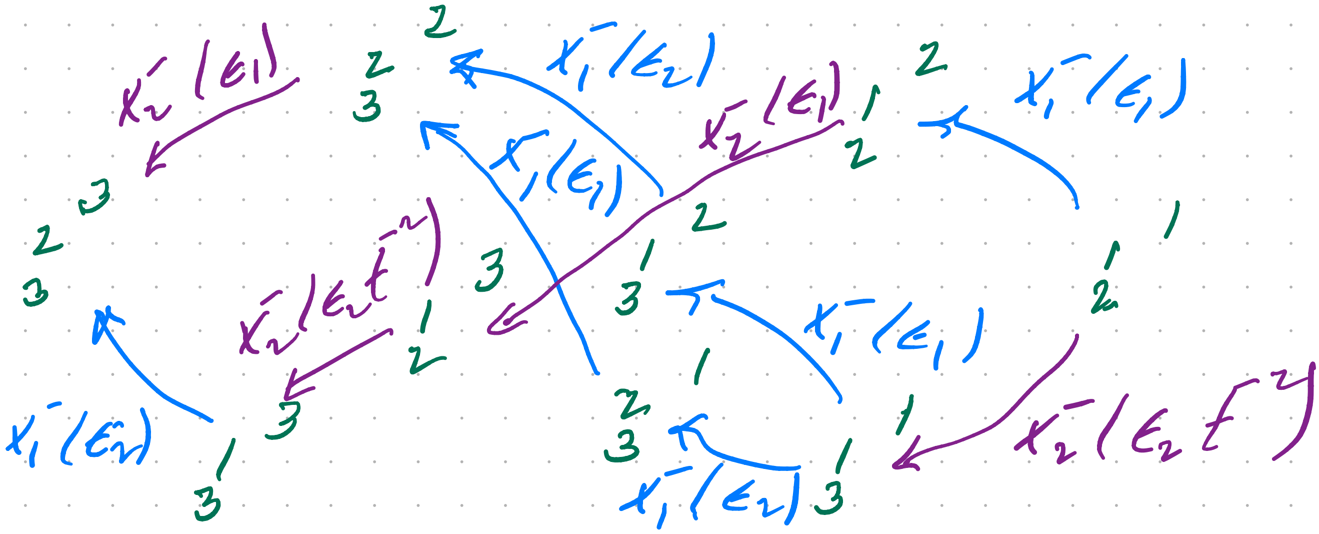
Let $a_1, a_2 \in \mathbb{C}^\times$.

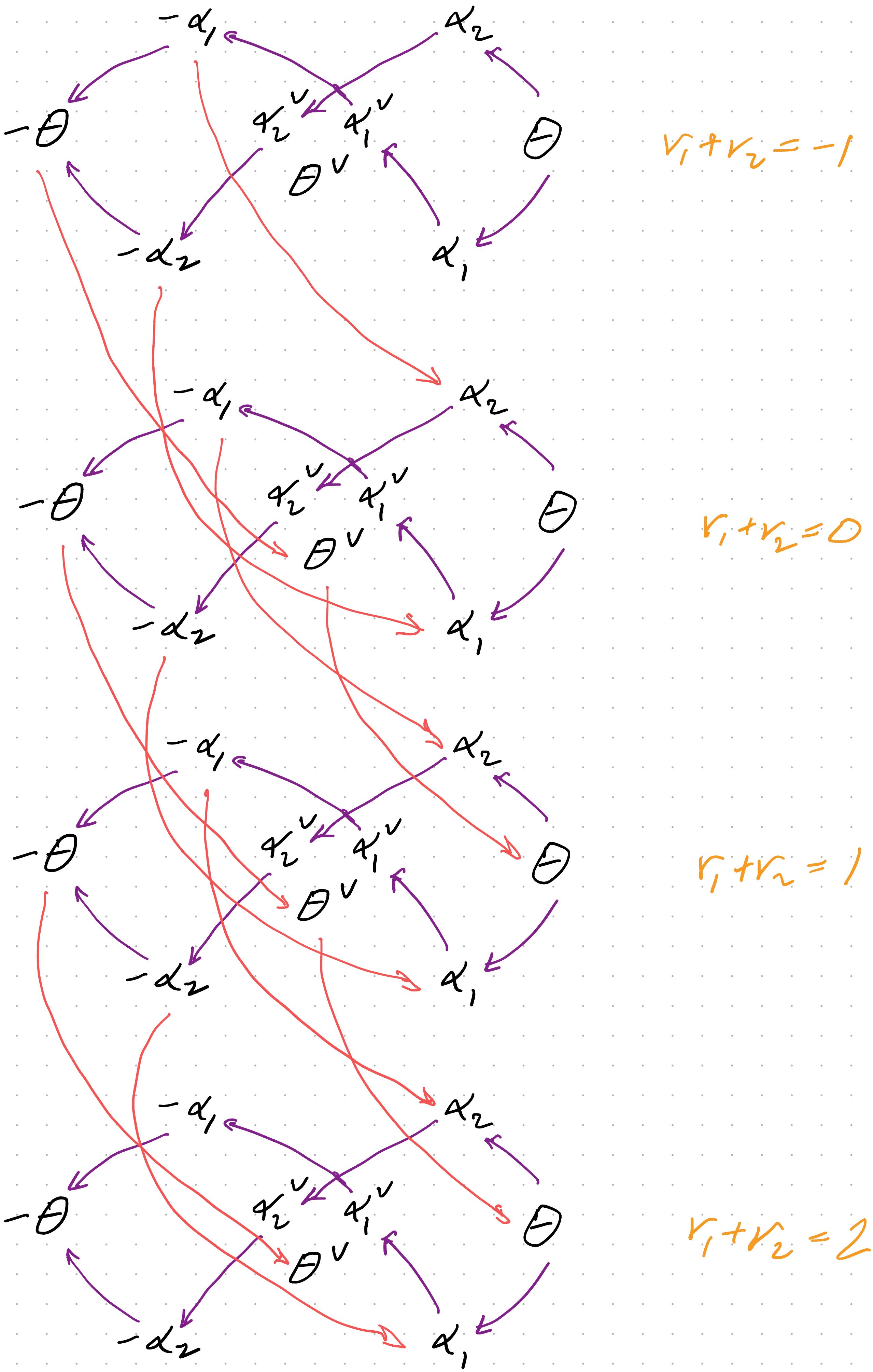
The <u>local Weyl module</u> at (a_1, a_2) is $L(\lambda)$ but with

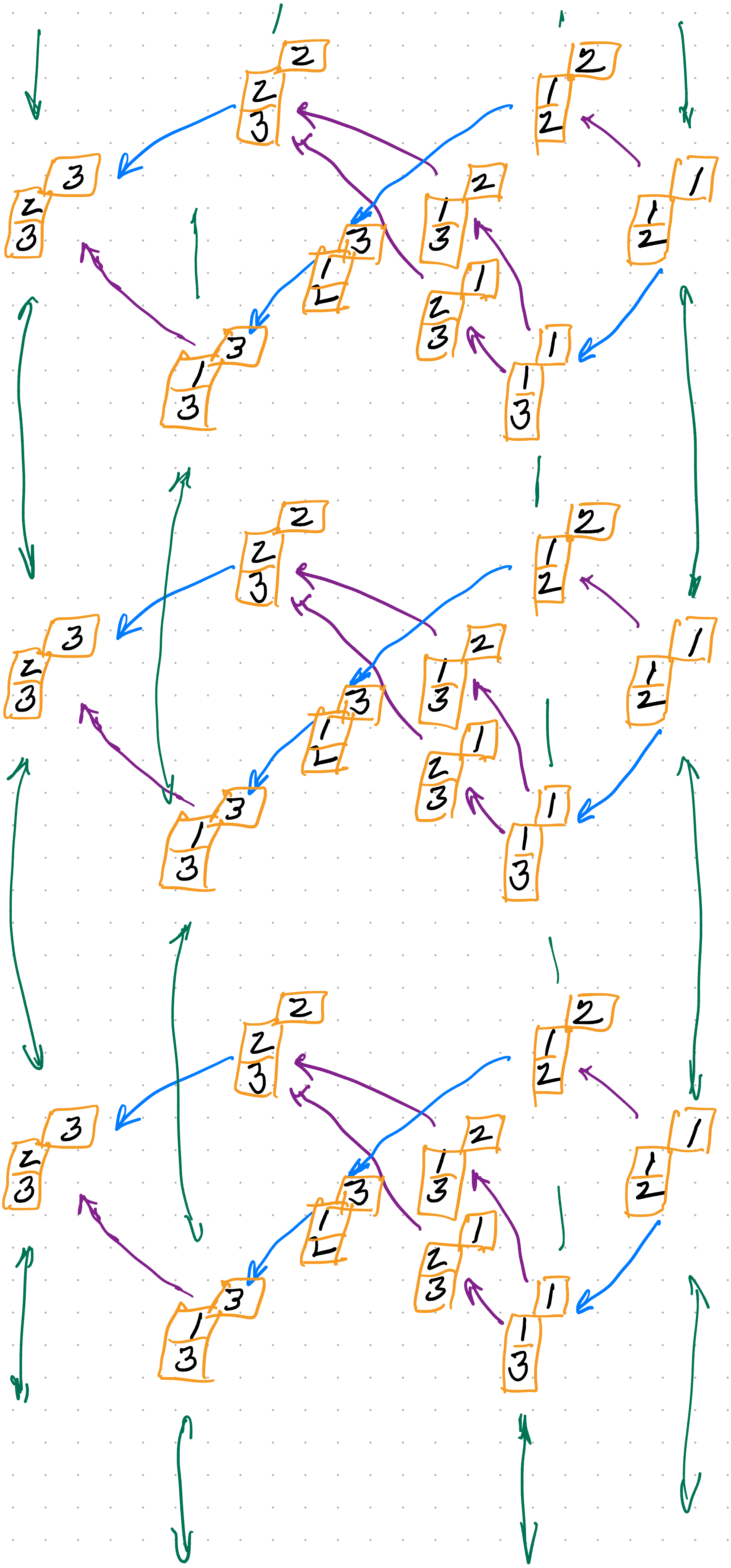
ε_1 specialized to a_1

ε_2 specialized to a_2.

<u>Loop action</u> "Eigenvectors" of $e_s^{(i)}$



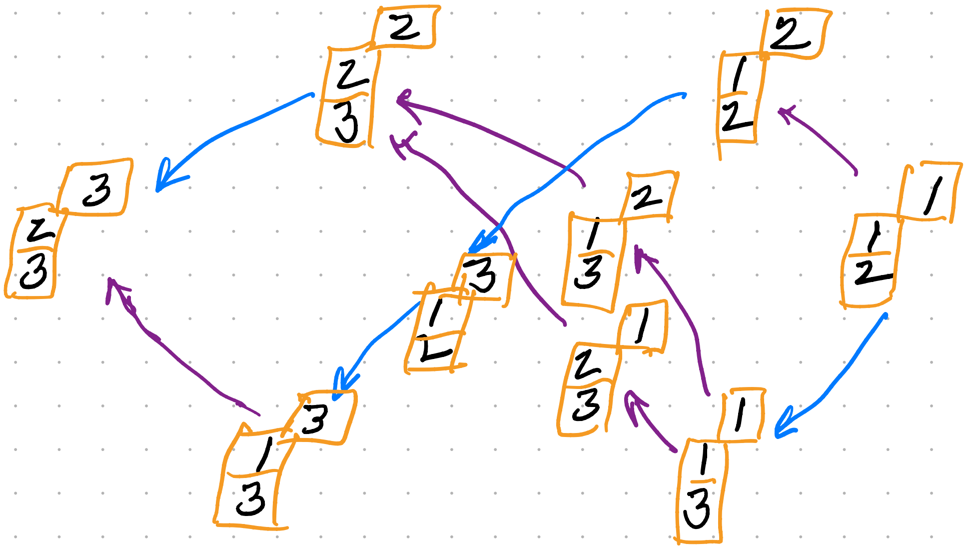

$-\alpha_1$
α_2
$-\theta$
$\alpha_2^\vee$
$\alpha_1^\vee$
$\theta^\vee$
θ
$-\alpha_2$
α_1
$r_1+r_2=-1$
$r_1+r_2=0$
$r_1+r_2=1$
$r_1+r_2=2$



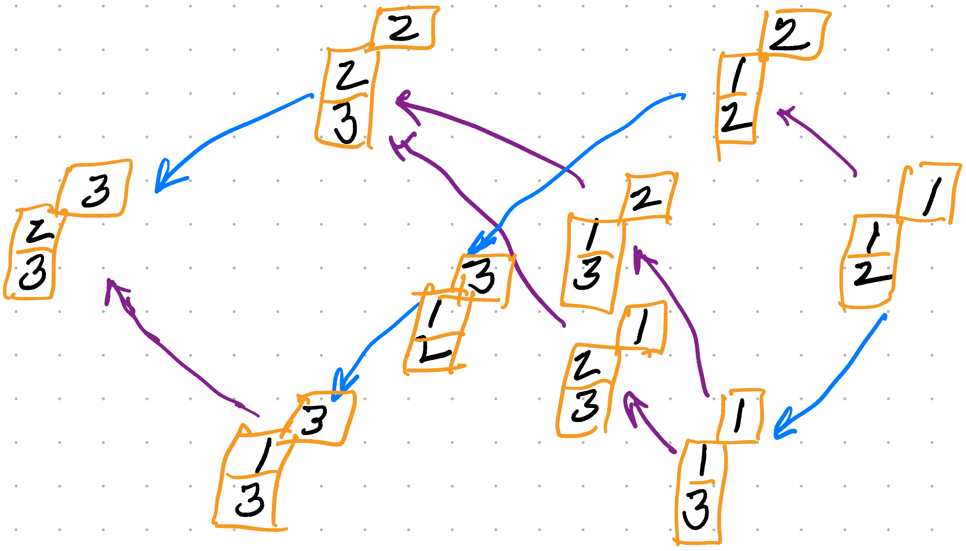
$r_1 + r_2 = -1$

$r_1 + r_2 = 0$

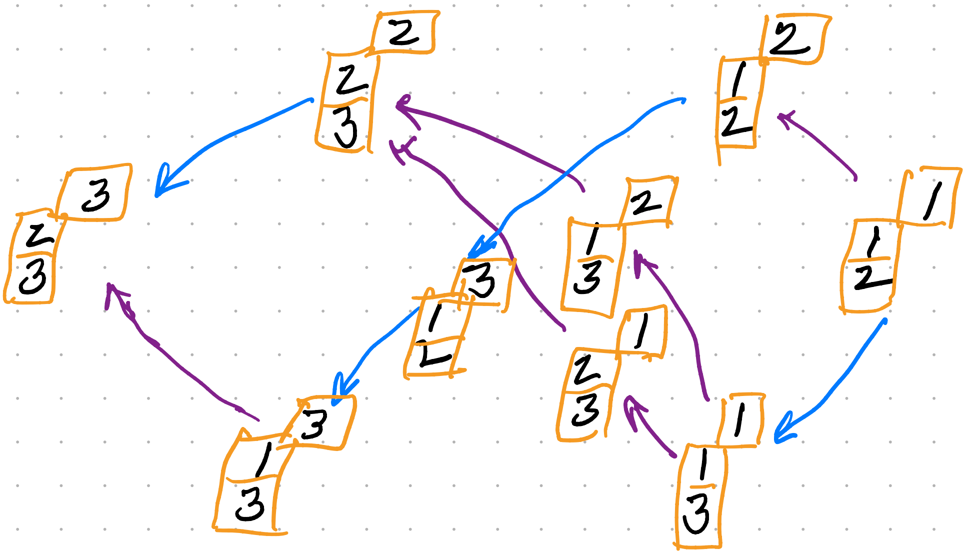
$r_1 + r_2 = 1$



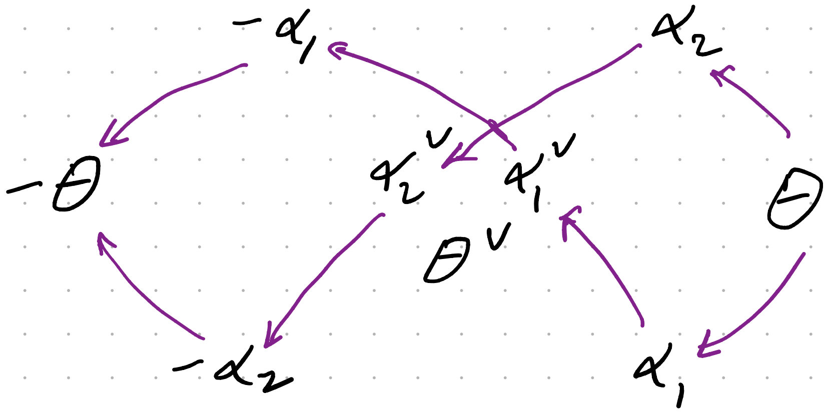
$r_1 + r_2 = -1$



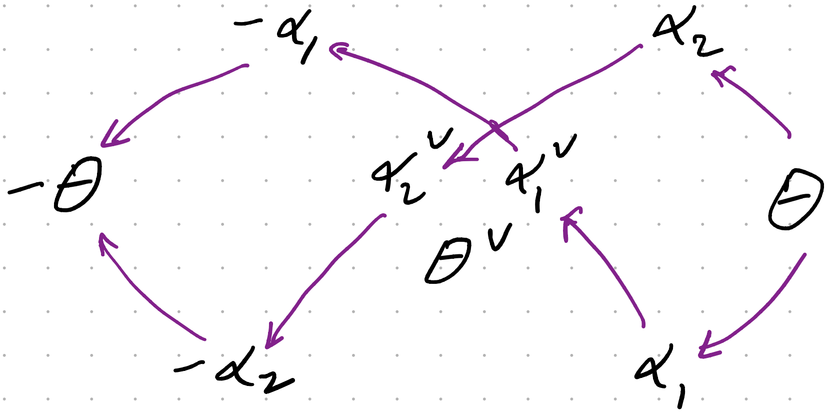
$r_1 + r_2 = 0$



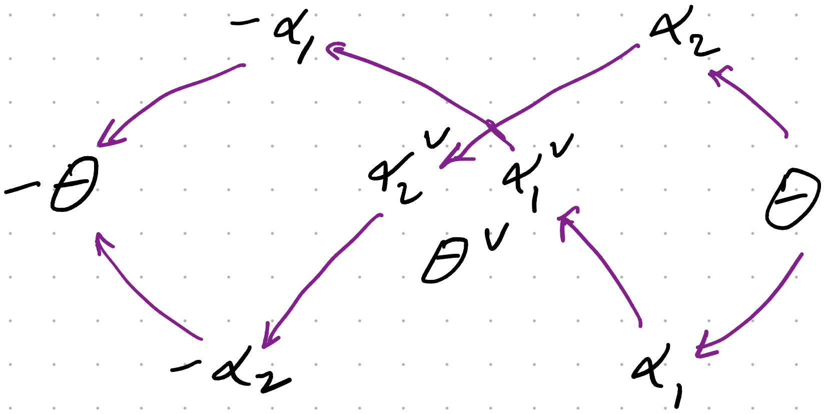
$r_1 + r_2 = 1$



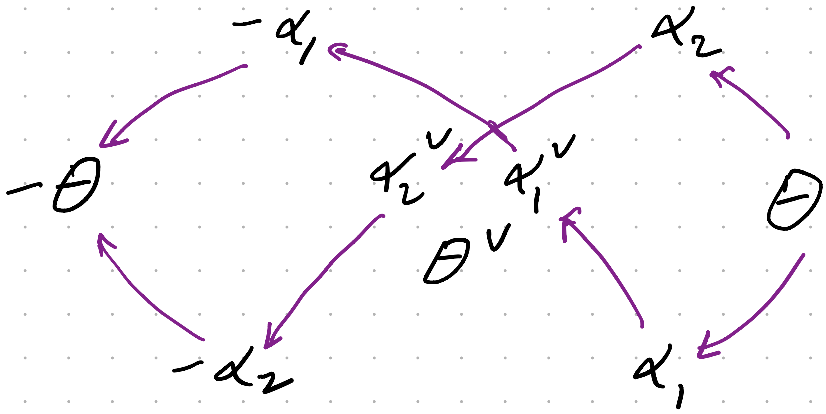
$r_1+r_2=-1$



$r_1+r_2=0$



$r_1+r_2=1$



$r_1+r_2=2$

The module $L(\omega_1)$ for $U_t\widehat{\mathfrak{sl}}_n$

Basis: $v_1\epsilon^r, \ldots, v_n\epsilon^r$ with $r \in \mathbb{Z}$

Let

$$v_{i+nr} = v_i\epsilon^r$$

so that v_k is defined for $k \in \mathbb{Z}$

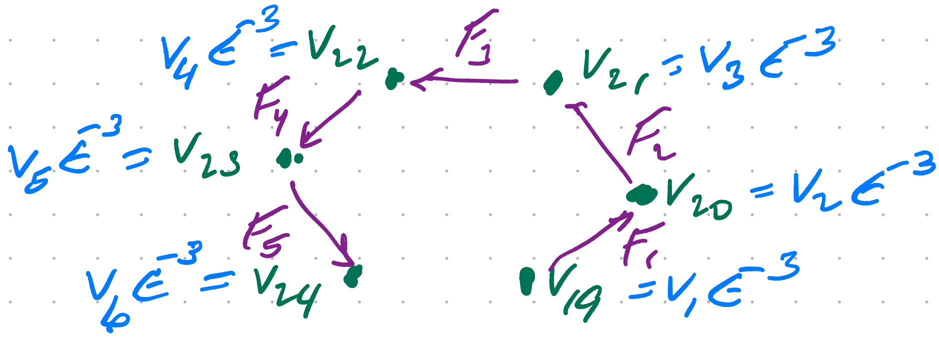
Kac-Moody action $k \in \mathbb{Z}$, $i \in \{1, \ldots, n\}$, $r \in \mathbb{Z}$

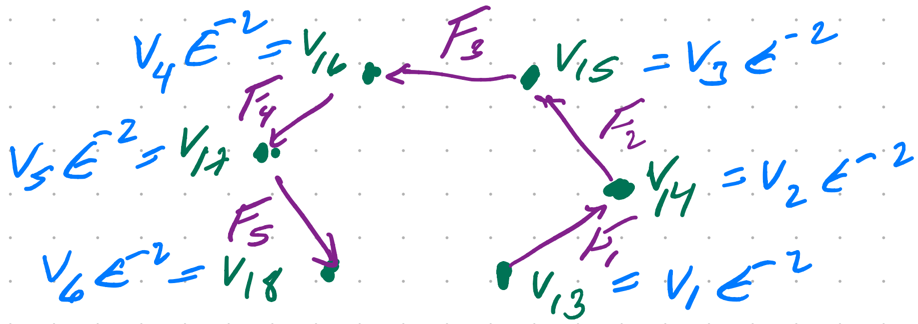
$$C^{\pm 1/2} v_k = v_k, \qquad D^{\pm 1}(v_i\epsilon^r) = t^{\pm r} v_i\epsilon^r$$

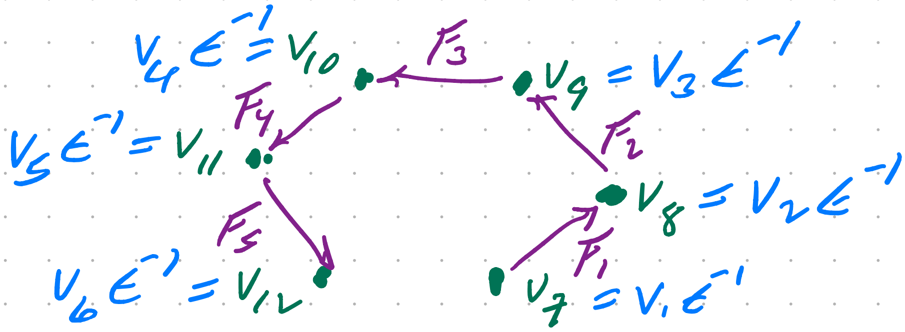
$$E_i v_k = \begin{cases} v_{k-1}, & \text{if } k = i+1 \bmod n, \\ 0, & \text{otherwise}, \end{cases}$$

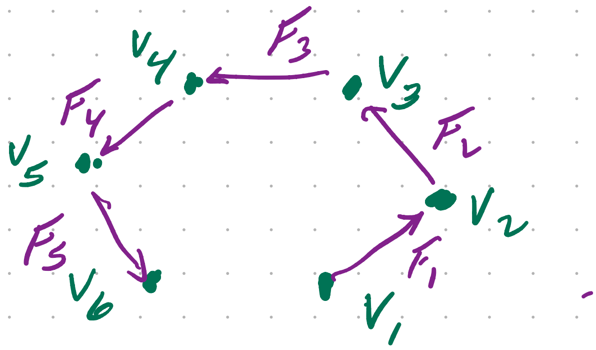
$$F_i v_k = \begin{cases} v_{k+1}, & \text{if } k = i \bmod n, \\ 0, & \text{otherwise} \end{cases}$$

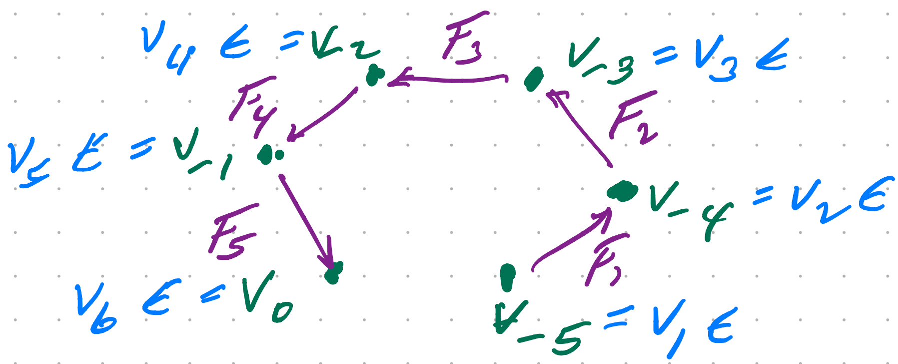
$$K_i v_k = \begin{cases} t v_k, & \text{if } k = i \bmod n \\ t^{-1} v_k, & \text{if } k = i+1 \bmod n \\ v_k, & \text{otherwise} \end{cases}$$

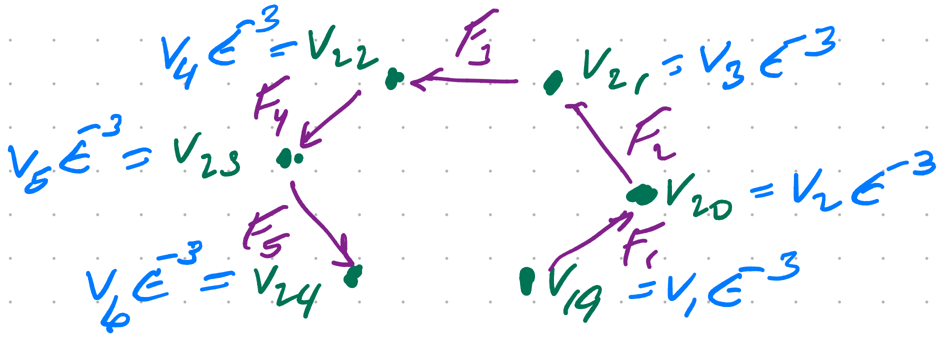

$V_4\epsilon^{-3} = V_{22}$
F_3
$V_{21} = V_3\epsilon^{-3}$
F_4
F_2
$V_5\epsilon^{-3} = V_{23}$
$V_{20} = V_2\epsilon^{-3}$
F_5
F_1
$V_6\epsilon^{-3} = V_{24}$
$V_{19} = V_1\epsilon^{-3}$

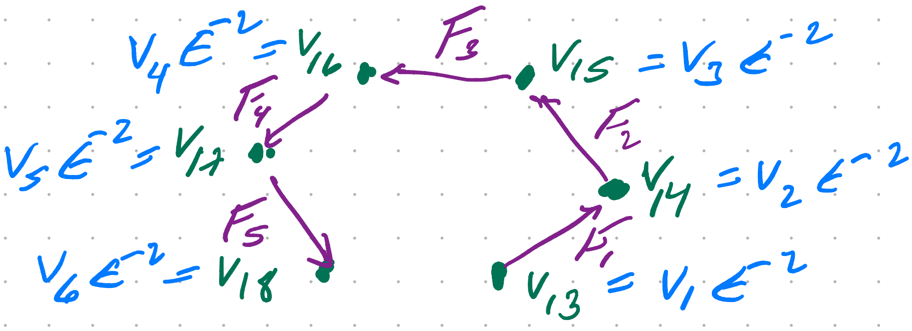

$V_4\epsilon^{-2} = V_{16}$
F_3
$V_{15} = V_3\epsilon^{-2}$
F_4
F_2
$V_5\epsilon^{-2} = V_{17}$
$V_{14} = V_2\epsilon^{-2}$
F_5
F_1
$V_6\epsilon^{-2} = V_{18}$
$V_{13} = V_1\epsilon^{-2}$

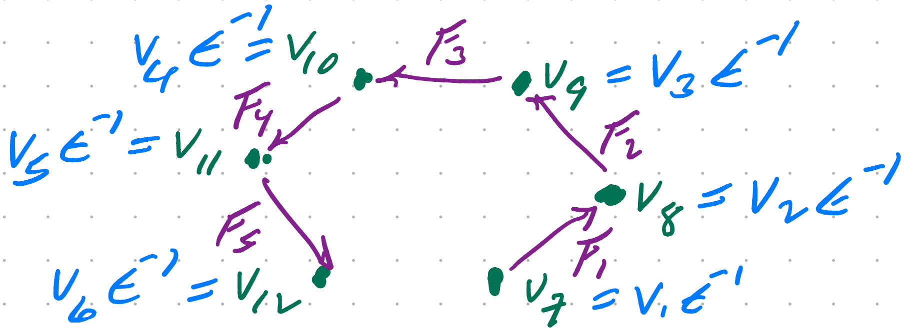

$V_4\epsilon^{-1} = V_{10}$
F_3
$V_9 = V_3\epsilon^{-1}$
F_4
F_2
$V_5\epsilon^{-1} = V_{11}$
$V_8 = V_2\epsilon^{-1}$
F_5
F_1
$V_6\epsilon^{-1} = V_{12}$
$V_7 = V_1\epsilon^{-1}$

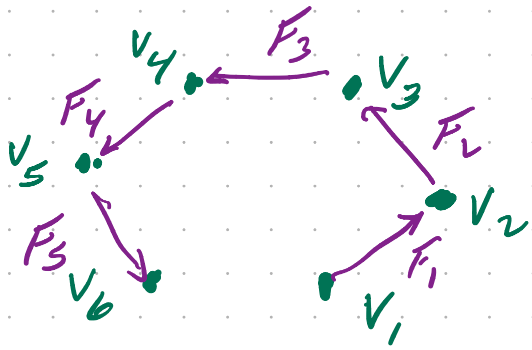

V_4
F_3
V_3
F_4
F_2
V_5
V_2
F_5
F_1
V_6
V_1

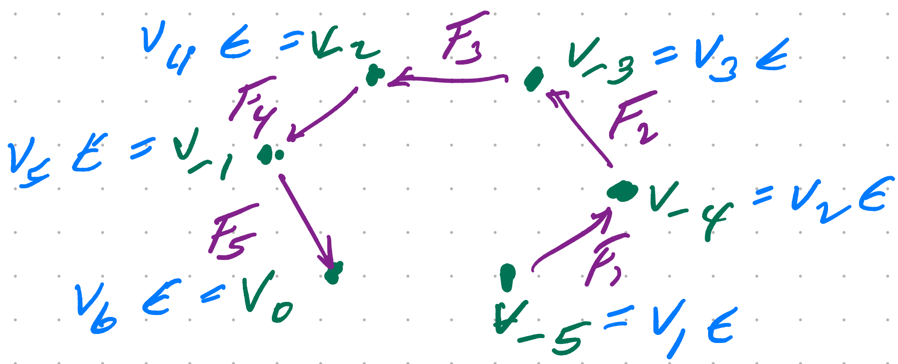

$V_4\epsilon = V_{-2}$
F_3
$V_{-3} = V_3\epsilon$
F_4
F_2
$V_5\epsilon = V_{-1}$
$V_{-4} = V_2\epsilon$
F_5
F_1
$V_6\epsilon = V_0$
$V_{-5} = V_1\epsilon$


$V_4 \epsilon^{-3} = V_{22}$
F_3
$V_{21} = V_3 \epsilon^{-3}$
F_4
F_2
$V_5 \epsilon^{-3} = V_{23}$
$V_{20} = V_2 \epsilon^{-3}$
F_5
F_1
$V_6 \epsilon^{-3} = V_{24}$
$V_{19} = V_1 \epsilon^{-3}$


$V_4 \epsilon^{-2} = V_{16}$
F_3
$V_{15} = V_3 \epsilon^{-2}$
F_4
F_2
$V_5 \epsilon^{-2} = V_{17}$
$V_{14} = V_2 \epsilon^{-2}$
F_5
F_1
$V_6 \epsilon^{-2} = V_{18}$
$V_{13} = V_1 \epsilon^{-2}$


$V_4 \epsilon^{-1} = V_{10}$
F_3
$V_9 = V_3 \epsilon^{-1}$
F_4
F_2
$V_5 \epsilon^{-1} = V_{11}$
$V_8 = V_2 \epsilon^{-1}$
F_5
F_1
$V_6 \epsilon^{-1} = V_{12}$
$V_7 = V_1 \epsilon^{-1}$


V_4
F_3
V_3
F_4
F_2
V_5
V_2
F_5
F_1
V_6
V_1


$V_4 \epsilon = V_{-2}$
F_3
$V_{-3} = V_3 \epsilon$
F_4
F_2
$V_5 \epsilon = V_{-1}$
$V_{-4} = V_2 \epsilon$
F_5
F_1
$V_6 \epsilon = V_0$
$V_{-5} = V_1 \epsilon$

The module $L(\omega_1)$ for $U_t\widehat{\mathfrak{sl}_n}$

Basis: $v_1\epsilon^r, \ldots, v_n\epsilon^r$ with $r\in\mathbb{Z}$

Loop action $j, i\in\{1,\ldots,n\}, r\in\mathbb{Z}, \ell\in\mathbb{Z}$

$C^{\pm 1/2} v_i\epsilon^r = v_i\epsilon^r, \quad D^{\pm 1}(v_i\epsilon^r) = t^{\pm r} v_i\epsilon^r$

$$x^+_{i,\ell} v_j\epsilon^r = \begin{cases} v_{j-1}\epsilon^{\ell+r}, & \text{if } j=i+1, \\ 0, & \text{otherwise,} \end{cases}$$

$$x^-_{i,\ell} v_j\epsilon^r = \begin{cases} v_{j+1}\epsilon^{\ell+r}, & \text{if } j=i \\ 0, & \text{otherwise} \end{cases}$$

$$q_{i,s} v_j\epsilon^r = \begin{cases} v_i\epsilon^{r+s}, & \text{if } j=i \\ -v_{i+1}\epsilon^{r+s}, & \text{if } j=i+1 \\ 0, & \text{otherwise.} \end{cases}$$

Kac-Moody presentation of $L(\lambda)$

Relations: $C^{\frac{1}{2}} u_{w\lambda} = u_{w\lambda}$,

$$K_i u_{w\lambda} = t^{\langle w\lambda, \alpha_i^\vee \rangle} u_{w\lambda},$$

$$E_i u_{w\lambda} = 0 \text{ and } F_i^{\langle w\lambda, \alpha_i^\vee \rangle} u_{w\lambda} = u_{s_i \cdot w\lambda}$$

$$\text{if } \langle w\lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}, \text{ and}$$

$$F_i u_{w\lambda} = 0 \text{ and } E_i^{-\langle w\lambda, \alpha_i^\vee \rangle} u_{w\lambda} = u_{s_i \cdot w\lambda}$$

$$\text{if } \langle w\lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{<0}.$$

Loop presentation of $L(\lambda)$

$$\lambda = m_1\omega_1 + \cdots + m_n\omega_n$$

Relations: $C^{\frac{1}{2}} u_\lambda = u_\lambda$,

$K_i \cdot u_\lambda = t^{m_i} u_\lambda$, $\quad x^+_{i,r} u_\lambda = 0$

$e^{(i)}_s u_\lambda = 0$, for $s \in \mathbb{Z}_{>m_i}$

$e^{(i)}_{-s} u_\lambda = 0$, for $s \in \mathbb{Z}_{>m_i}$

where $e^{(i)}_s$ for $s \in \mathbb{Z}_{\neq 0}$ are defined by

$$\exp\left(\sum_{r\in\mathbb{Z}_{>0}} \frac{P^{(i)}_r}{[r]} z^r\right) = 1 + \sum_{s\in\mathbb{Z}_{>0}} e^{(i)}_s z^s$$

$$\exp\left(\sum_{r\in\mathbb{Z}_{>0}} \frac{P^{(i)}_{-r}}{[r]} z^{-r}\right) = 1 + \sum_{s\in\mathbb{Z}_{>0}} e^{(i)}_{-s} z^{-s}$$

The module $L(\omega_1+\omega_2)$ for $U_t(\hat{sl}_3)$

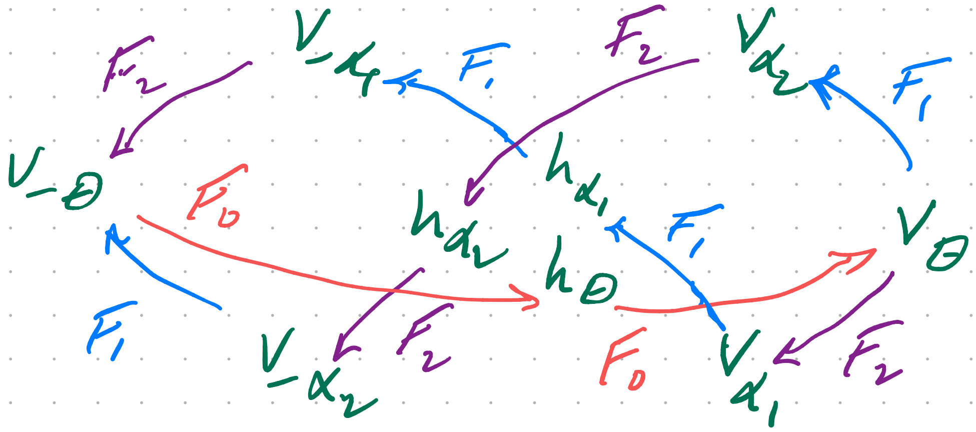
Basis: $\epsilon_1^{r_1}\epsilon_2^{r_2}v_\theta$, $\epsilon_1^{r_1}\epsilon_2^{r_2}v_{\alpha_1}$, $\epsilon_1^{r_1}\epsilon_2^{r_2}v_{-\alpha_2}$,

$\epsilon_1^{r_1}\epsilon_2^{r_2}v_{\alpha_2}$, $\epsilon_1^{r_1}\epsilon_2^{r}v_{-\alpha_1}$, $\epsilon_1^{r_1}\epsilon_2^{r_2}v_{-\theta}$,

$\epsilon_1^{r_1}\epsilon_2^{r_2}h_\theta$, $\epsilon_1^{r_1}\epsilon_2^{r_2}h_{\alpha_1}$, $\epsilon_1^{r_1}\epsilon_2^{r_2}h_{\alpha_2}$

with $r_1, r_2 \in \mathbb{Z}$.

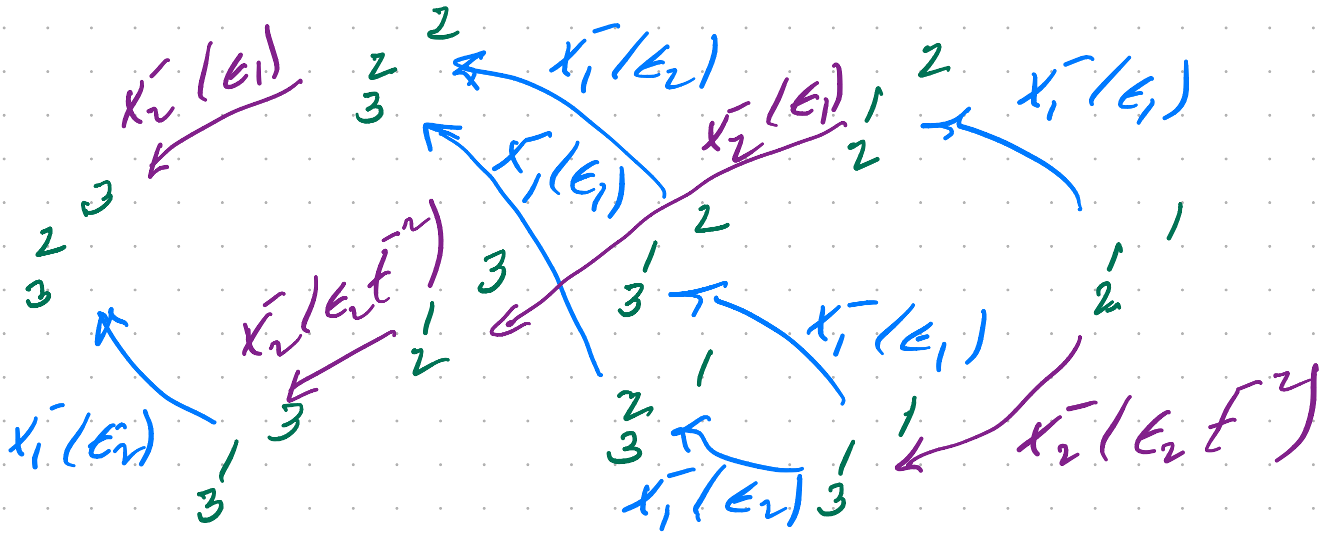
Kac-Moody action



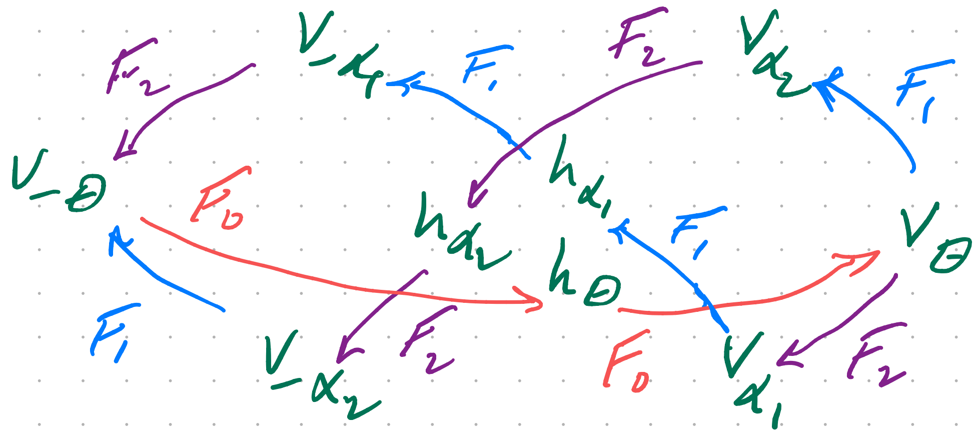
after setting $\epsilon_1=1$ and $\epsilon_2=1$.

Loop action

Eigenvectors of $q_+^{(i)}, q_-^{(i)}$



Kac-Moody action



after setting $\epsilon_1 = 1$ and $\epsilon_2 = 1$.

The module $L(\omega_1 + \omega_2)$ for $U_t(\hat{sl}_3)$

Basis:

$$\epsilon_1^{r_1}\epsilon_2^{r_2} v_{\alpha_1},\ \epsilon_1^{r_1}\epsilon_2^{r_2} v_{-\alpha_2},$$

$$\epsilon_1^{r_1}\epsilon_2^{r_2} v_{\theta},\qquad \epsilon_1^{r_1}\epsilon_2^{r_2} v_{-\theta},$$

$$\epsilon_1^{r_1}\epsilon_2^{r_2} v_{\alpha_2},\ \epsilon_1^{r_1}\epsilon_2^{r} v_{-\alpha_1},$$

$$\epsilon_1^{r_1}\epsilon_2^{r_2} h_{\theta},\ \epsilon_1^{r_1}\epsilon_2^{r_2} h_{\alpha_1},\ \epsilon_1^{r_1}\epsilon_2^{r_2} h_{\alpha_2}$$

with $r_1, r_2 \in \mathbb{Z}$.