Examples in affine Combinatorial Representation Theory

Talk 3: ASEP and transfer matrices

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multispecies ASEP on a circle

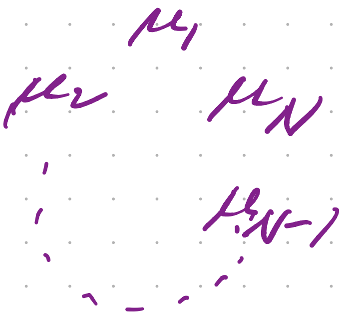
Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \{1, \ldots, n\}^N$

$$\mathcal{I}_\lambda = \mathbb{C}\text{-span}\{ |\mu\rangle \mid \mu \in S_N \lambda \}$$

the span of symbols $|\mu\rangle$ indexed by rearrangements μ of λ.

Let $t \in \mathbb{R}_{[0,1]}$.

The <u>Markov matrix</u> $M^\lambda : \mathcal{I}_\lambda \to \mathcal{I}_\lambda$



- Choose a position (uniformly)
- If $\mu_i > \mu_{i+1}$ then switch.
- If $\mu_i < \mu_{i+1}$ then flip a coin (probability of heads t). If heads then switch.

Example $\lambda = (2,1,0)$

$$
\begin{array}{c|cccccc}
 & \overset{2}{10} & \overset{1}{20} & \overset{2}{01} & \overset{1}{02} & \overset{0}{21} & \overset{0}{12} \\
\hline
\overset{2}{10} & \circledast & t/n & t/n & 0 & 0 & 1/n \\
\overset{1}{20} & 1/n & \circledast & 0 & t/n & 1/n & 0 \\
\overset{2}{01} & 1/n & 0 & \circledast & 1/n & 1/n & 0 \\
\overset{1}{02} & 0 & 1/n & t/n & \circledast & 0 & t/n \\
\overset{0}{21} & 0 & t/n & 1/n & 0 & \circledast & t/n \\
\overset{0}{12} & t/n & 0 & 0 & 1/n & t/n & \circledast
\end{array}
$$

where $\circledast$ is computed to make the sum of entries in a column equal to 1.

Note: $M^\lambda = \sum_{i=1}^{N} \frac{1}{n} \check{R}'_{i,i+1}$ where

$$
\check{R}'_{i,i+1} = \begin{array}{c|cc}
 & \mu_i \mu_{i+1} & \mu_{i+1} \mu_i \\
\hline
\mu_i \mu_{i+1} & \circledast & t \\
\mu_{i+1} \mu_i & 1 & \circledast
\end{array}
$$

with indices mod N.

The <u>stationary distribution</u> is

$\pi \in I_\lambda$ with $M\pi = \pi$.

Let
$$E_\lambda(x_1, \dots, x_N; q, t)$$
be the nonsymmetric Macdonald polynomial.

Let $\mu \in S_N \lambda$ and let $z_\mu \in S_n$ be minimal length such that
$$z_\mu \lambda = \mu.$$

Let T_i be the operator coming from the DAHA action on polynomials and
$$T_{z_\mu} = T_{j_1} \cdots T_{j_k} \text{ if } z_\mu = s_{j_1} \cdots s_{j_k}$$
is a reduced word.

The permuted basement Macdonald polynomial

$$f_\mu(x_1, \dots x_n; q, t) = t^{\frac{1}{2}\ell(z_\mu)} T_{z_\mu} E_\lambda$$

Note: The symmetric Macdonald polynomial is

$$P_\lambda = \sum_{\mu \in S_N \lambda} f_\mu$$

Theorem The stationary distribution of M^λ is

$$\pi = \sum_{\mu \in S_N \lambda} f_\mu(1, \dots, 1; 1, t) |\mu\rangle.$$

Theorem Let

$$\Psi = \sum_{\mu \in S_N \lambda} f_\mu(x_1, \dots, x_N; q, t) |\mu\rangle$$

Then Ψ is an eigenvector of

$$T_N(x_1, \dots, x_N; q, t),$$

the inhomogeneous transfer matrix with q-twisted boundary condition.

Proof idea:

(1) $T_N(x_1, \dots, x_N; q, t)$

$$= \bigoplus_{\lambda \text{ decreasing}} T_N^{\lambda}(x_1, \dots, x_N; q, t).$$

(2) Since

$$T_N(x_1, \dots, x_N; q, t) = R_{01}\left(\frac{x_1}{q}\right) R_{02}\left(\frac{x_2}{q}\right) \cdots R_{0N}\left(\frac{x_N}{q}\right)$$

with

$$R_{0j}(z) = \begin{pmatrix} \frac{t-1}{t-z} & \frac{t(1-z)}{t-z} \\ \frac{1-z}{t-z} & \frac{(t-1)z}{t-z} \end{pmatrix}$$

then

$$T_N(x_1,\dots,x_N;q,t)$$

$$= c_0 + \left(\frac{x_1}{q}-1\right)\check{R}_{12} + \left(\frac{x_1}{q}-1\right)\check{R}_{23} + \cdots + \left(\frac{x_1}{q}-1\right)\check{R}_{N1}$$

+ higher degree terms in

$$\left(\frac{x_1}{q}-1\right), \dots, \left(\frac{x_N}{q}-1\right).$$

(3) $T_N(z,z,\dots z;1,t)$

$= c_0 + M^\lambda(z-1) +$ higher degree terms in $z-1$ //.

Algebraic Bethe ansatz

(following Takhtadjan-Faddeev 1979).

Let $U = U_t\mathfrak{g}$ the quantum affine algebra.

Let V and A be level 0 integrable modules.

$\rho : U \to End(V)$ and $\pi : U \to End(A)$

and $N \in \mathbb{Z}_{>0}$.

Hamitonian $H_N : V^{\otimes N} \to V^{\otimes N}$

$$T_N(z) = C_0 + H_N(z-1) + \text{higher degree terms in } z-1.$$

Partition function $Z_{M\times N}(z) \in \mathbb{C}$.

$$Z_{M\times N}(z) = Tr_{V^{\otimes N}}\left(T_N(z)^M\right).$$

Transfer matrix $T_N(z) : V^{\otimes N} \to V^{\otimes N}$

$$T_N(z) = \mathrm{Tr}_A\left(\mathcal{T}_N(z)\right).$$

Monodromy matrix $\mathcal{T}_N : A \otimes V^{\otimes N} \to A \otimes V^{\otimes N}$

$$\mathcal{T}_N(z) = (\pi \otimes \rho^{\otimes N})(\mathcal{R}(z))$$

L-matrix $L(z) : A \otimes V \longrightarrow A \otimes V$

$$L(z) = (\pi \otimes \rho)(\mathcal{R}(z)).$$

R-matrix $\mathcal{R}(z) \in \mathcal{U} \otimes \mathcal{U}.$

$$\mathcal{R}(z) = (\tau_z \otimes id)(\mathcal{R}).$$

The initial data is

$(\mathcal{U}, \mathcal{R}, \tau_z)$ a pseudo quasitriangular Hopf algebra

universal R-matrix ($\mathcal{R}$)

automorphisms of $\mathcal{U}$ indexed by $z \in \mathbb{C}^\times$. (τ_z)

Schur-Weyl duality for $U_t \widehat{\mathfrak{sl}}_n$

$$V = L(\omega_1) = \mathbb{C}^n[t, t^{-1}]$$
$$= \mathrm{span}\{t^r v_1, \ldots, t^r v_n \mid r \in \mathbb{Z}\}$$

Then $V^{\otimes N} = V(z_1) \otimes \cdots \otimes V(z_N)$

$$= \mathrm{span}\left\{ z_1^{r_1} v_{i_1} \otimes \cdots \otimes z_N^{r_N} v_{i_N} \;\middle|\; \begin{matrix} r_1, \ldots, r_N \in \mathbb{Z} \\ i_1, \ldots, i_N \in \{1, \ldots, n\} \end{matrix} \right\}$$

$$= \mathrm{span}\left\{ z_1^{r_1} \cdots z_N^{r_N} \otimes v_{\mu_1} \otimes \cdots \otimes v_{\mu_N} \;\middle|\; \begin{matrix} r_1, \ldots, r_N \in \mathbb{Z} \\ \mu_1, \ldots, \mu_N \in \{1, \ldots, n\} \end{matrix} \right\}$$

$$= \mathrm{span}\left\{ z_1^{r_1} \cdots z_N^{r_N} |\mu\rangle \;\middle|\; \begin{matrix} r_1, \ldots, r_N \in \mathbb{Z} \\ \mu_1, \ldots, \mu_N \in \{1, \ldots, n\} \end{matrix} \right\}$$

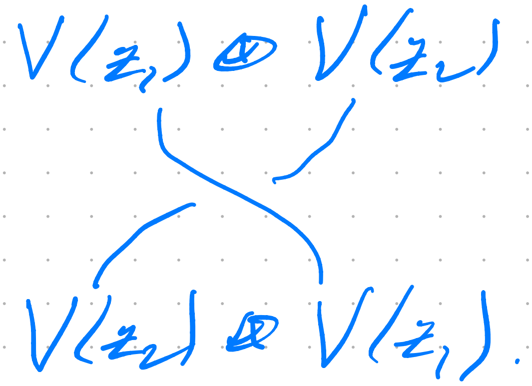
$$= \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes \left(\bigoplus_{\lambda \text{ decreasing}} I_\lambda \right)$$

where $I_\lambda = \mathrm{span}\{ |\mu\rangle \mid \mu \in S_N \lambda \}$.

The R-matrix gives a U_t-module morphism

$$\check{R} : V \otimes V \longrightarrow V \otimes V$$

Pictorially

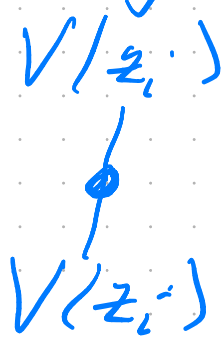


The ~~t~~ gives U_t-module morphisms

$Y_i : V(z_i) \otimes V(z_i)$

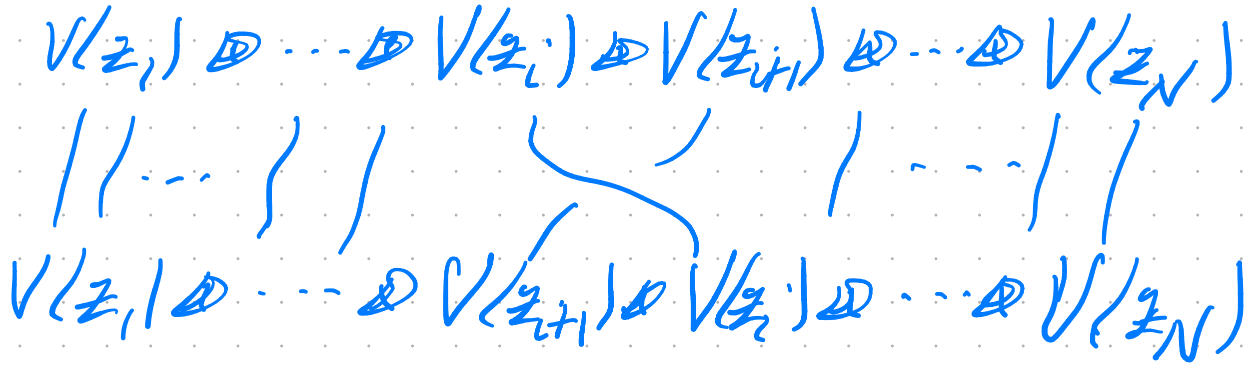
$z_i^r v_j \longmapsto z_i^{r+1} v_j$

Pictorially

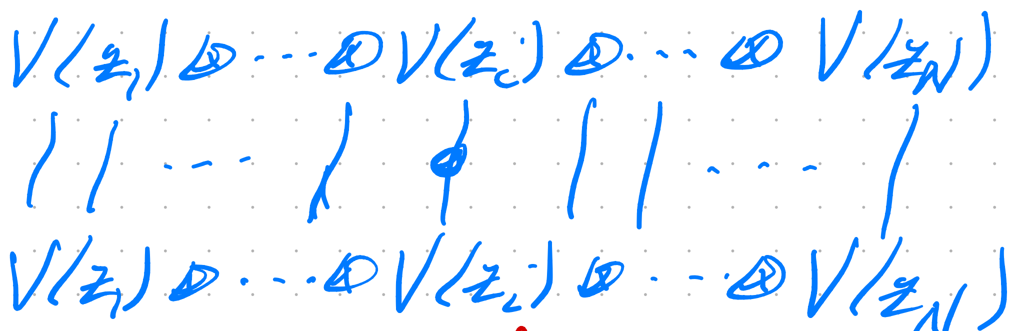


Together $\check{R}_1, \ldots, \check{R}_{N-1}$ and $Y_1, \ldots, Y_N$

$\check{R}_i =$



$Y_i =$



give an action of the affine Hecke algebra on $V^{\otimes N}$.

This is what is used to make a connection to Macdonald polynomials.

The module $L(w_1)$ for $U_t \widehat{\mathfrak{sl}}_n$

Basis: $v_1\epsilon^r, \ldots, v_n\epsilon^r$ with $r \in \mathbb{Z}$

Let
$$v_{i+nr} = v_i \epsilon^r$$
so that v_k is defined for $k \in \mathbb{Z}$

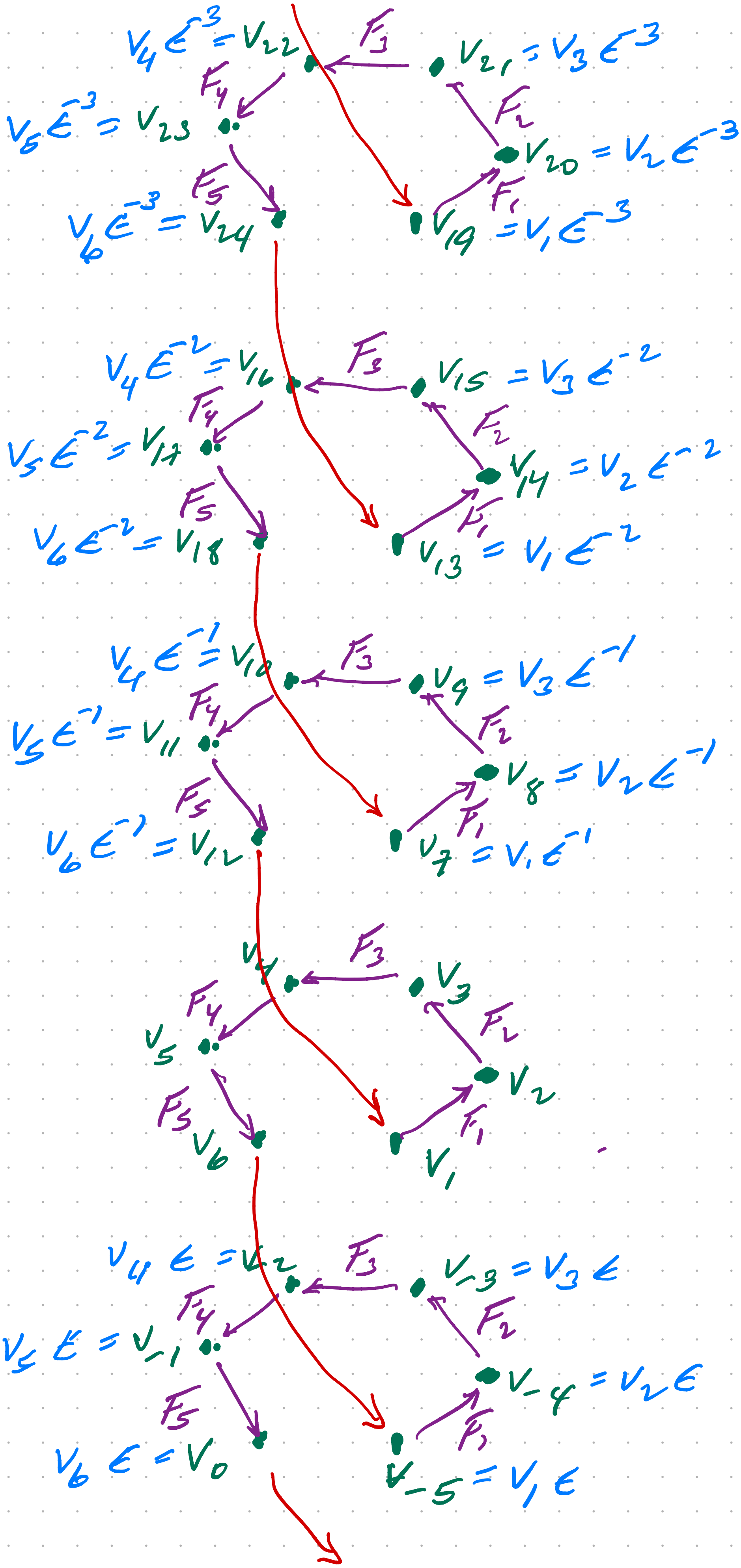
Kac-Moody action $k \in \mathbb{Z}$, $i \in \{1, \ldots, n\}$, $r \in \mathbb{Z}$

$$C^{\pm 1/2} v_k = v_k, \qquad D^{\pm 1}(v_i \epsilon^r) = t^{\pm r} v_i \epsilon^r$$

$$E_i v_k = \begin{cases} v_{k-1}, & \text{if } k = i+1 \bmod n, \\ 0, & \text{otherwise,} \end{cases}$$

$$F_i v_k = \begin{cases} v_{k+1}, & \text{if } k = i \bmod n, \\ 0, & \text{otherwise} \end{cases}$$

$$K_i v_k = \begin{cases} t v_k, & \text{if } k = i \bmod n \\ t^{-1} v_k, & \text{if } k = i+1 \bmod n \\ v_k, & \text{otherwise} \end{cases}$$



$V_4\epsilon^{-3} = V_{22}$
F_3
$V_{21} = V_3\epsilon^{-3}$
F_4
$V_5\epsilon^{-3} = V_{23}$
F_2
$V_{20} = V_2\epsilon^{-3}$
F_5
F_1
$V_6\epsilon^{-3} = V_{24}$
$V_{19} = V_1\epsilon^{-3}$
$V_4\epsilon^{-2} = V_{16}$
F_3
$V_{15} = V_3\epsilon^{-2}$
F_4
$V_5\epsilon^{-2} = V_{17}$
F_2
$V_{14} = V_2\epsilon^{-2}$
F_5
F_1
$V_6\epsilon^{-2} = V_{18}$
$V_{13} = V_1\epsilon^{-2}$
$V_4\epsilon^{-1} = V_{10}$
F_3
$V_9 = V_3\epsilon^{-1}$
F_4
$V_5\epsilon^{-1} = V_{11}$
F_2
$V_8 = V_2\epsilon^{-1}$
F_5
F_1
$V_6\epsilon^{-1} = V_{12}$
$V_7 = V_1\epsilon^{-1}$
V_4
F_3
V_3
F_4
V_5
F_2
V_2
F_5
F_1
V_6
V_1
$V_4\epsilon = V_{-2}$
F_3
$V_{-3} = V_3\epsilon$
F_4
$V_5\epsilon = V_{-1}$
F_2
$V_{-4} = V_2\epsilon$
F_5
F_1
$V_6\epsilon = V_0$
$V_{-5} = V_1\epsilon$

The module $L(\omega_1)$ for $U_t\hat{sl}_n$

Basis: $v_1\epsilon^r, \dots, v_n\epsilon^r$ with $r \in \mathbb{Z}$

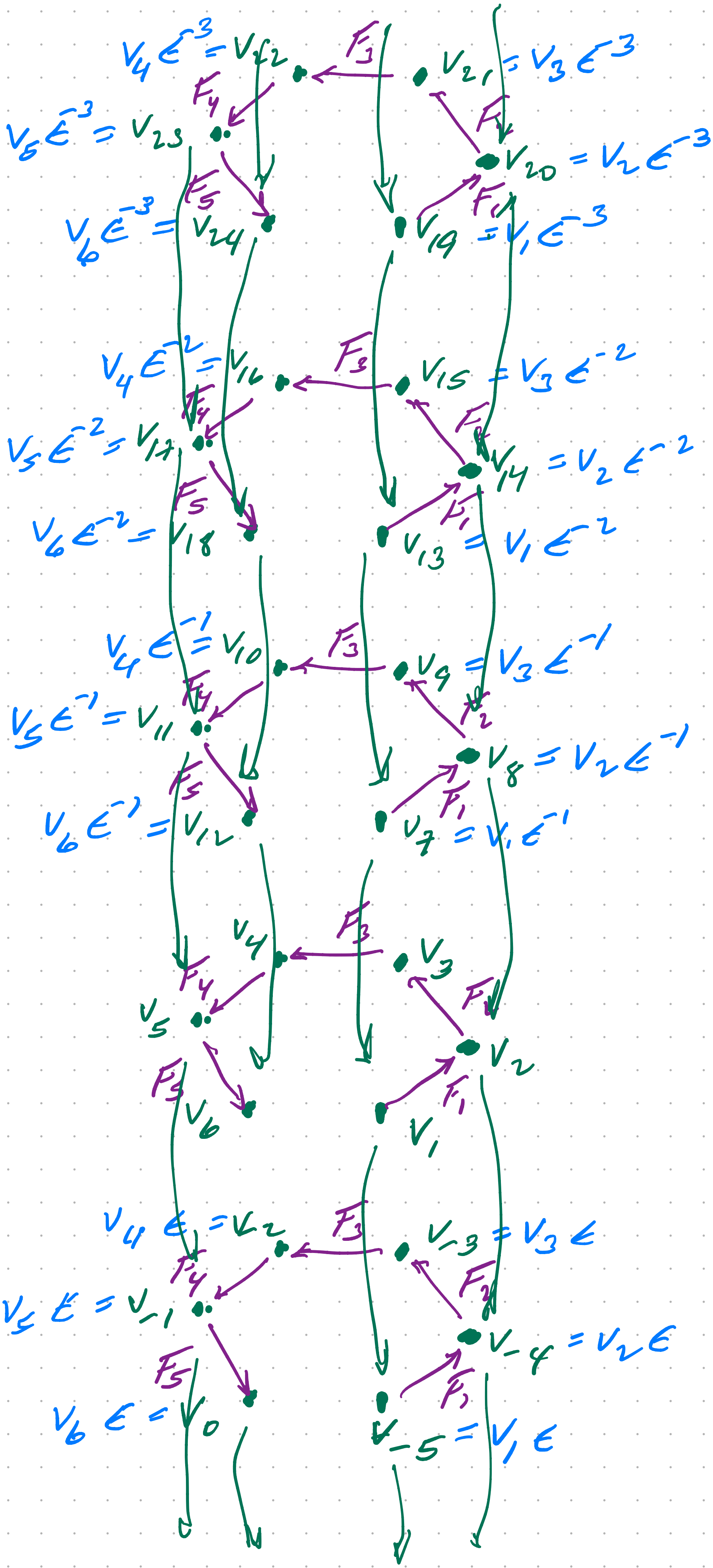
Loop action $j, i \in \{1, \dots, n\}, r \in \mathbb{Z}, \ell \in \mathbb{Z}$

$C^{\pm 1/2} v_i \epsilon^r = v_i \epsilon^r, \quad D^{\pm 1}(v_i \epsilon^r) = t^{\pm r} v_i \epsilon^r$

$$x^+_{i,\ell} v_j \epsilon^r = \begin{cases} v_{j-1}\epsilon^{\ell+r}, & \text{if } j = i+1, \\ 0, & \text{otherwise}, \end{cases}$$

$$x^-_{i,\ell} v_j \epsilon^r = \begin{cases} v_{j+1}\epsilon^{\ell+r}, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

$$q_{i,s} v_j \epsilon^r = \begin{cases} v_i \epsilon^{r+s}, & \text{if } j = i \\ -v_{i+1}\epsilon^{r+s}, & \text{if } j = i+1 \\ 0, & \text{otherwise}. \end{cases}$$



V4 ε^-3 = V22
V21 = V3 ε^-3
F3
F4
V5 ε^-3 = V23
F2
V20 = V2 ε^-3
F5
F1
V6 ε^-3 = V24
V19 = V1 ε^-3
V4 ε^-2 = V16
V15 = V3 ε^-2
V5 ε^-2 = V17
V14 = V2 ε^-2
V6 ε^-2 = V18
V13 = V1 ε^-2
V4 ε^-1 = V10
V9 = V3 ε^-1
V5 ε^-1 = V11
V8 = V2 ε^-1
V6 ε^-1 = V12
V7 = V1 ε^-1
V4
V3
V5
V2
V6
V1
V4 ε = V-2
V-3 = V3 ε
V5 ε = V-1
V-4 = V2 ε
V6 ε = V0
V-5 = V1 ε