13.03.2021

Macdonald Ch. VI §6   Talk Prep. ①

The symmetric Macdonald polynomial

$$P_\lambda = P_\lambda(x;q,t) = P_\lambda(x_1,\dots,x_n;q,t)$$

The Schur function

$$P_\lambda(x;t,t) = s_\lambda = s_\lambda(x_1,\dots,x_n) = \mathrm{char}(L(\lambda))$$

where $L(\lambda)$ is the irred. integrable rep. of $GL_n(\mathbb{C})$ of highest weight λ.

dimension $s_\lambda(1,1,\dots,1) = \dim(L(\lambda))$

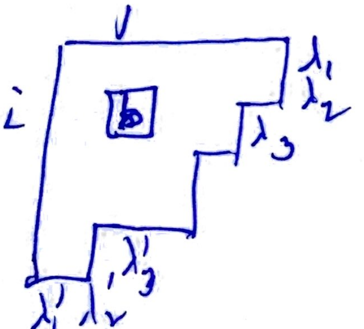
quantum dimension $s_\lambda(1,t,t^2,\dots,t^{n-1}) = \mathrm{qdim}(L(\lambda))$

elliptic dimension? $P_\lambda(1,t,t^2,\dots,t^{n-1};q,t) = \mathrm{edim}(L(\lambda))$

From Ch. I

$$s_\lambda(1,t,t^2,\dots,t^{n-1}) = \prod_{b\in\lambda} \frac{1-t^{n+c(b)}}{1-t^{h(b)}}$$

$$s_\lambda(1,1,\dots,1) = \lim_{t\to 1} s_\lambda(1,t,t^2,\dots,t^{n-1}) = \prod_{b\in\lambda}\frac{n+c(b)}{h(b)}$$



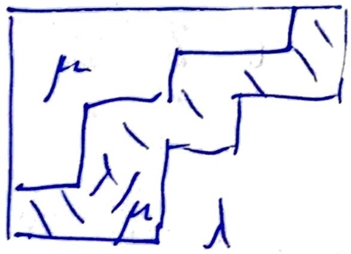
$c(b) = j-i$ content of b

$h(b) = \lambda_i - i + \lambda_j' - j + 1$ hook length at b.

Define

$$d_\lambda = P_\lambda(1, t, t^2, \dots, t^{n-1}; q, t), \qquad b_\lambda = \frac{1}{\langle P_\lambda, P_\lambda \rangle}$$

$$P_\mu e_r = \sum_\lambda \psi'_{\lambda/\mu} P_\lambda$$



<u>theorem</u>

(a)
$$d_\lambda = t^{n(\lambda)} \prod_{i<j} \frac{(q^{\lambda_i-\lambda_j} t^{j-i}; q)_\infty \, (t^{j-i+1}; q)_\infty}{(q^{\lambda_i-\lambda_j} t^{j-i+1}; q)_\infty \, (t^{j-i}; q)_\infty} \qquad (6.11)$$

$$= t^{n(\lambda)} \prod_{b \in \lambda} \frac{1 - q^{\mathrm{coarm}_\lambda(b)} t^{n - \mathrm{coleg}_\lambda(b)}}{1 - q^{\mathrm{arm}_\lambda(b)} t^{\mathrm{leg}_\lambda(b)+1}} \qquad (6.11')$$

(b)
$$b_\lambda = \prod_{i<j} \frac{(q^{\lambda_i-\lambda_j} t^{j-i+1}; q)_\infty \, (q^{\lambda_i-\lambda_j+1} t^{j-i-1}; q)_\infty}{(q^{\lambda_i-\lambda_j} t^{j-i}; q)_\infty \, (q^{\lambda_i-\lambda_j+1} t^{j-i}; q)_\infty} \qquad \text{§4 Ex. 11(a)}$$

$$= \prod_{b \in \lambda} \frac{1 - q^{\mathrm{arm}_\lambda(b)} t^{\mathrm{leg}_\lambda(b)+1}}{1 - q^{\mathrm{arm}_\lambda(b)+1} t^{\mathrm{leg}_\lambda(b)}} \qquad (6.19)$$

(c)
$$\psi'_{\lambda/\mu} = \prod_{i<j} \frac{(1 - q^{\mu_i-\mu_j} t^{j-i+1})(1 - q^{\lambda_i-\lambda_j} t^{j-i+1})}{(1 - q^{\mu_i-\mu_j} t^{j-i})(1 - q^{\lambda_i-\lambda_j} t^{j-i})} \qquad (6.13)$$

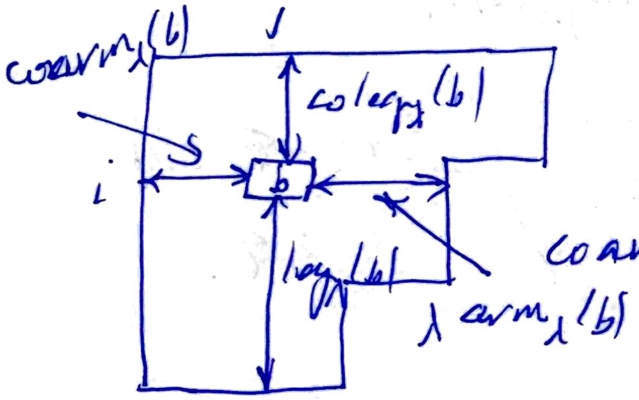
$$= \prod_{\substack{s=(r,c) \in \lambda \\ \lambda'_c \neq \mu'_c,\ \lambda_r = \mu_r}} \frac{(1 - q^{\mathrm{arm}_\lambda(b)} t^{\mathrm{leg}_\lambda(b)+1})(1 - q^{\mathrm{arm}_\mu(b)+1} t^{\mathrm{leg}_\mu(b)})}{(1 - q^{\mathrm{arm}_\lambda(b)+1} t^{\mathrm{leg}_\lambda(b)})(1 - q^{\mathrm{arm}_\mu(b)} t^{\mathrm{leg}_\mu(b)+1})} \qquad (6.23)$$

if λ/μ is a vertical strip, and $\psi'_{\lambda/\mu} = 0$ otherwise
(does not contain $\square\square$)

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$.

$$n(\lambda) = \sum_{i=1}^{n} (i-1)\lambda_i \qquad n\left(\begin{array}{|c|c|c|c|c|} \hline 0&0&0&0&0 \\ \hline 1&1&1&1& \\ \hline 2&2&2&2& \\ \hline 3&3&3&& \\ \hline 4&4&&& \\ \hline \end{array}\right) = 8+9+8+4 = 29$$

$$(u;q) = (1-u)(1-uq)(1-uq^2)\cdots$$



$$\mathrm{coleg}_\lambda(b) = j-1$$

$$\mathrm{coarm}_\lambda(b) = i-1 \qquad \mathrm{arm}_\lambda(b) = \lambda_i - i$$

$$\mathrm{leg}_\lambda(b) = \lambda'_j - j$$

Note
$$u + c(b) = \mathrm{coarm}_\lambda(b) + n - \mathrm{coleg}_\lambda(b)$$
$$h(b) = \mathrm{arm}_\lambda(b) + \mathrm{leg}_\lambda(b) + 1$$

The theorem says

$$t^{n(\lambda)} \prod_{i<j} \frac{1-t^{\lambda_i - \lambda_j + j - i}}{1-t^{j-i}} = t^{n(\lambda)} \prod_{b\in\lambda} \frac{1-t^{u+c(b)}}{1-t^{h(b)}}$$

Prove that

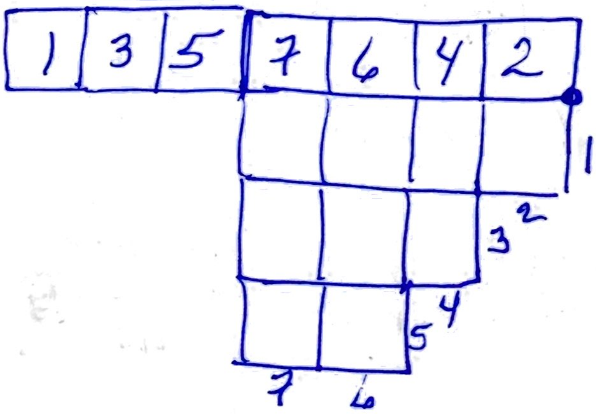
$$t^{29}\,\frac{\begin{matrix}1-t^2 & 1-t^3 & 1-t^5 & 1-t^7\\ & 1-t & 1-t^3 & 1-t^5\\ & & 1-t^2 & 1-t^4\\ & & & 1-t^2\end{matrix}}{\begin{matrix}1-t & 1-t^2 & 1-t^3 & 1-t^4\\ & 1-t & 1-t^2 & 1-t^3\\ & & 1-t & 1-t^2\\ & & & 1-t\end{matrix}} = t^{29}\,\frac{\begin{matrix}1-t^5 & 1-t^6 & 1-t^7 & 1-t^8 & 1-t^9\\ 1-t^4 & 1-t^5 & 1-t^6 & 1-t^7\\ 1-t^3 & 1-t^4 & 1-t^5 & 1-t^6\\ 1-t^2 & 1-t^3 & 1-t^4\\ 1-t & 1-t^2\end{matrix}}{\begin{matrix}1-t^9 & 1-t^8 & 1-t^6 & 1-t^4 & 1-t\\ 1-t^7 & 1-t^6 & 1-t^4 & 1-t^2\\ 1-t^6 & 1-t^5 & 1-t^3 & 1-t\\ 1-t^4 & 1-t^3 & 1-t\\ 1-t^2 & 1-t\end{matrix}}$$

To show

$$\begin{array}{rrrr|rrrrr} 2 & 3 & 5 & 7 & 9 & 8 & 6 & 4 & 1\\ & 1 & 3 & 5 & 7 & 6 & 4 & 2 & \\ & & 2 & 4 & 6 & 5 & 3 & 1 & \\ & & & 2 & 4 & 3 & 1 & & \\ & & & & 2 & 1 & & & \end{array} = \begin{array}{rrrr|rrrrr} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ & & 1 & 2 & 3 & 4 & 5 & 6 & \\ & & & 1 & 2 & 3 & 4 & & \\ & & & & 1 & 2 & & & \end{array}$$

To show: $\boxed{1\;3\;5 \mid 7\;6\;4\;2} = \boxed{1\;2\;3\;4\;5\;6\;7}$

Proof:



To show:

$$\prod_{i<j}\prod_{r\geq 0}^{\lambda_i-\lambda_j-1}\frac{1-t^{j-i+1}q^r}{1-t^{j-i}q^r} = \prod_{s\in\lambda}\frac{1-q^{\mathrm{coarm}_\lambda(s)}t^{n-\mathrm{coleg}_\lambda(s)}}{1-q^{\mathrm{arm}_\lambda(s)}t^{\mathrm{leg}_\lambda(s)+1}}$$

To show:

Left side, numerator:

$1-q^2t^5$ $1-qt^5$ $1-t^5$	$1-qt^4$ $1-t^4$	$1-t^3$	$1-t^2$
	$1-qt^4$ $1-t^4$	$1-t^3$	
		$1-qt^3$ $1-t^3$	$1-t^2$
			$1-t^2$

Right side, numerator:

$1-q^0t^5$	$1-qt^5$	$1-q^2t^5$	$1-q^3t^5$	$1-q^4t^5$
$1-q^0t^4$	$1-qt^4$	$1-q^2t^4$	$1-q^3t^4$	
$1-q^0t^3$	$1-qt^3$	$1-q^2t^3$	$1-q^3t^3$	
$1-q^0t^2$	$1-qt^2$	$1-q^2t^2$		
$1-q^0t$	$1-qt$			

Left side, denominator:

$1-q^2t^4$ $1-qt^4$ $1-t^4$	$1-qt^3$ $1-t^3$	$1-t^2$	$1-t$
	$1-qt^3$ $1-t^3$	$1-t^2$	
		$1-qt^2$ $1-t^2$	$1-t$
			$1-t$

=

Right side, denominator:

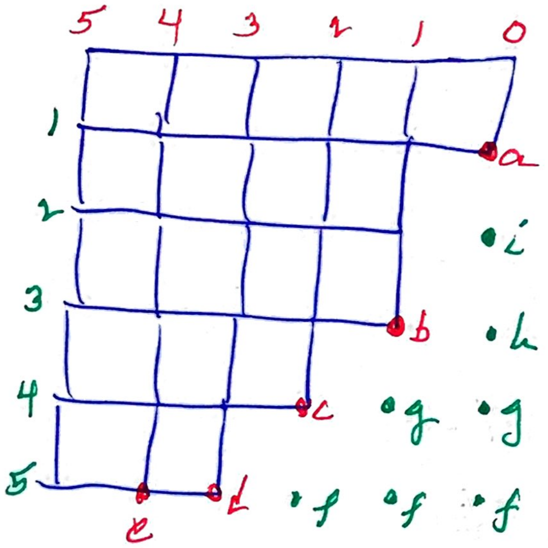
$1-q^4t^5$	$1-q^3t^5$	$1-q^2t^4$	$1-qt^3$	$1-q^0t$
$1-q^3t^4$	$1-q^2t^4$	$1-qt^3$	$1-q^0t^2$	
$1-q^3t^3$	$1-q^2t^3$	$1-qt^2$	$1-q^0t$	
$1-q^2t^2$	$1-qt^2$	$1-q^0t$		
$1-qt$	$1-q^0t$			

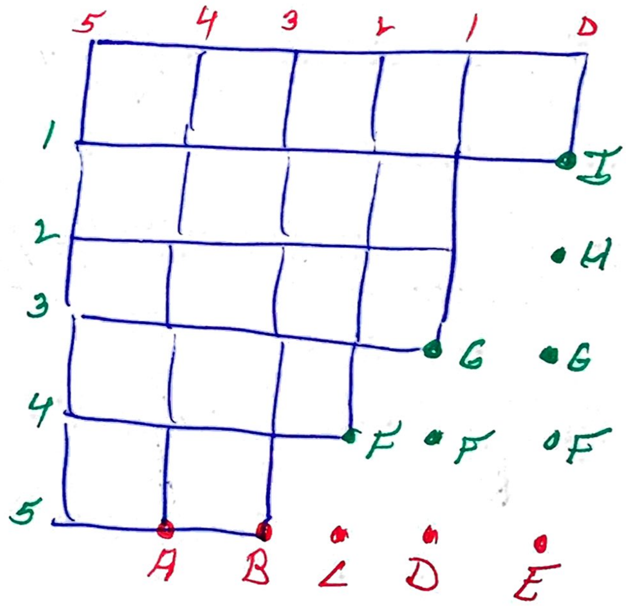
To show:

f	g	h	i	e	d	c	b	a
(2,5) (1,5) (0,5)	(1,4) (0,4)	(0,3)	(0,2)	(4,5)	(3,5)	(2,4)	(1,3)	(0,1)

=

F	G	H	I	E	D	C	B	A
(2,4) (1,4) (0,4)	(1,3) (0,3)	(0,2)	(0,1)	(0,5)	(1,5)	(2,5)	(3,5)	(4,5)





Do the other rows similarly.