21.03.2011
Talk prep. ①

The Y-evaluation homomorphism

$ev_\mu^\rho : \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}] \longrightarrow \mathbb{C}[q^{\pm 1}, t^{\pm 1}]$ is given by

$$ev_\mu^\rho (Y_i) = q^{\mu_i} t^{n-i}$$

so that

$$D_n^r (P_\mu) = ev_\mu^\rho (e_r) P_\mu, \quad \text{where}$$

$e_r = e_r(Y_1, \ldots, Y_n)$ is the elementary symmetric function.

The X-evaluation homomorphism

$ev_0^{\rho^\vee} : \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \longrightarrow \mathbb{C}[q^{\pm 1}, t^{\pm 1}]$ is given by

$$ev_0^{\rho^\vee} (X_i) = t^{n-i}$$

so that

$$ev_0^{\rho^\vee} (P_\mu) = P_\mu (1, t, t^2, \ldots, t^{n-1}; q, t)$$

is the principal specialization of P_μ.

Define

$$\tilde{P}_\mu = \frac{1}{ev_0^{\rho^\vee}(P_\mu)} P_\mu$$

("normalized by dimension")

Theorem (Pieri rule)

Let $A_I(y_1,\dots,y_n) = t^{\frac{1}{2}r(r-1)}\left(\prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j}\right)$

then

$$e_r(x_1,\dots,x_n)\tilde{P}_\mu = \sum_{\substack{I\subseteq\{1,\dots,n\}\\ |I|=r}} ev^\rho_\mu(A_I)\tilde{P}_{\mu+I}$$

Proof sketch (post Cherednik)

As operators on $\mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]^{S_n}$

$$e_r(x_1,\dots,x_n) = \sum_{\substack{I\subseteq\{1,\dots,n\}\\ |I|=r}} \xi^I A_I(y_1,\dots,y_n)$$

where $\xi^I : \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]^{S_n} \to \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]^{S_n}$ is defined by

$$\xi^I\tilde{P}_\mu = \tilde{P}_{\mu+I}$$

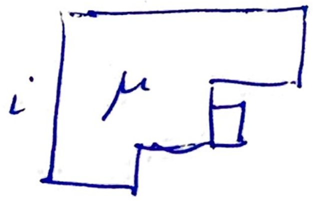
(if $I=\{i_1,\dots,i_r\}$ and $\varepsilon_j = (0,\dots,0,1,0,\dots,0)$ with 1 in the j^{th} spot then

$$\mu+I = \mu+\varepsilon_{i_1}+\dots+\varepsilon_{i_r}. \quad //.$$

Example: r=1

$$e_1(x_1,\dots,x_n) = x_1+\dots+x_n.$$

$$e_1(x_1,\dots x_n)\,\tilde P_\mu = \sum_{i=1}^{n} t^0\, ev^0_\mu\left(\prod_{j\neq i}\frac{ty_i-y_j}{y_i-y_j}\right)\tilde P_{\mu+\varepsilon_i}$$



If $\mu_i=\mu_{i-1}$ then

$$ev^0_\mu(ty_i-y_{i-1}) = tq^{\mu_i}t^{n-i} - q^{\mu_{i-1}}t^{n-(i-1)} = q^{\mu_i}t^{n-i}(t-t)$$
$$=0.$$

Remark. Define

$$A_I(x_1,\dots,x_n) = t^{\frac{1}{2}r(r-1)}\left(\prod_{\substack{i\in I\\ j\notin I}}\frac{tx_i-x_j}{x_i-x_j}\right)$$

Then

$$D_n^r = e_r(Y_1,\dots,Y_n) = \sum_{\substack{I\subseteq\{1,\dots,n\}\\ |I|=r}} A_I(x_1,\dots,x_n)\left(\prod_{i\in I} t_{q,x_i}\right)$$

or

$$D_n^r = \sum_{\substack{I\subseteq\{1,\dots,n\}\\ |I|=r}} A_I(x_1,\dots,x_n)\, y_I$$

and

$$e_r(x_1,\dots,x_n) = \sum_{\substack{I\subseteq\{1,\dots,n\}\\ |I|=r}} \xi^I A_I(Y_1,\dots,Y_n).$$

21.03.2021 Talk prep (4)

Setting up duality

The DAWG (double affine Weyl group) $\tilde{W}$ is the group generated by $q, x_1, \dots, x_n, y_1, \dots, y_n$ and S_n with relations $q \in Z(\tilde{W})$

$$x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i, \quad x_i y_i = q y_i x_i$$

$$w x_i = x_{w(i)} x_i, \quad w y_i = y_{w(i)} w, \quad x_i y_j = y_j x_i$$

for $i, j \in \{1, \dots, n\}$ with $i \neq j$ and $w \in S_n$

Let

$\varepsilon_i = (0, \dots, 0, 1, 0, \dots 0)^t$ with 1 in ith spot

$\varepsilon_j^\vee = (0, \dots, 0, 1, 0, \dots 0)$ with 1 in jth spot.

For $\mu = (\mu_1, \dots, \mu_n)^t \in \mathbb{Z}^n$ and $\lambda^\vee = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ let

$x_\mu = x_{\mu_1 \varepsilon_1 + \dots + \mu_n \varepsilon_n} = x_1^{\mu_1} \dots x_n^{\mu_n}$ and

$y_{\lambda^\vee} = y_{\lambda_1 \varepsilon_1^\vee + \dots + \lambda_n \varepsilon_n^\vee} = (y_1)^{\lambda_1} \dots (y_n)^{\lambda_n}$.

Then

$$y_{\lambda^\vee} x_\mu = q^{\langle \lambda^\vee, \mu \rangle} x_\mu y_{\lambda^\vee}$$

where the $\mathbb{Z}$-bilinear form $\langle , \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ is given by

$$\langle \varepsilon_i^\vee, \varepsilon_j \rangle = \delta_{ij}.$$

Heisenberg, Polynomials and shift ops

The subgroup

$$\mathrm{Heis} = \{ x_\mu y_{\lambda^\vee} \mid \mu \in \mathbb{Z}^n, \lambda^\vee \in \mathbb{Z}^n \}$$

is the Heisenberg group

The polynomial representation of $\tilde{W}$ is

$$\mathbb{C}[\tilde{W}]\mathbf{1} = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\mathbf{1} \quad \text{with}$$

$$y_{\lambda^\vee}\mathbf{1} = \mathbf{1} \quad \text{and} \quad w\mathbf{1} = \mathbf{1}$$

for $\lambda^\vee \in \mathbb{Z}^n$ and $w \in S_n$. Let $x^\mu = x_\mu \mathbf{1}$.

Then

$$x_\mu x^\nu = x^{\mu+\nu} \quad \text{and} \quad y_{\lambda^\vee} x^\nu = q^{\langle \lambda^\vee, \nu \rangle} x^\nu$$

So $y_{\lambda^\vee} : \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and

$$y_i = y_{\varepsilon_i} = T_{q, x_i} \quad \text{and}$$

$$y_I = y_{\varepsilon_{i_1} + \cdots + \varepsilon_{i_r}} = \left(\prod_{i \in I} T_{q, x_i} \right) \quad \text{if } I = \{i_1, \ldots, i_r\}.$$

Difference operators

Let W_Y be the subgroup of $\tilde{W}$ given by

$$W_Y = \{ q^r w y_{\lambda^\vee} \mid r \in \mathbb{Z}, w \in S_n, \lambda^\vee \in \mathbb{Z}^n \}.$$

Let $\mathbb{C}(X)[W_Y]$ be the group algebra of W_Y with coefficients in $\mathbb{C}(x_1, \dots, x_n)$

i.e.

$$\mathbb{C}(X)[W_Y] = \left\{ \sum_{\substack{r \in \mathbb{Z} \\ w \in S_n \\ \lambda^\vee \in \mathbb{Z}}} f_{r,w,\lambda^\vee}(x_1, \dots, x_n)\, q^r w y_{\lambda^\vee} \quad \text{with} \quad f_{r,w,\lambda^\vee} \in \mathbb{C}(x_1, \dots, x_n) \right\}$$

A *difference operator* is $D \in \mathbb{C}(X)[W_Y]$ such that

if $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ then $Df \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Examples of difference operators

$$D_n^r = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=r}} t^{\frac{1}{2}r(r-1)} \left(\prod_{\substack{j \notin I \\ i \in I}} \frac{tx_i - x_j}{x_i - x_j} \right) y_I$$

Let

$$g = y_1 s_1 \cdots s_{n-1}$$

$$T_i = \frac{(1-t)x_{i+1}}{x_i - x_{i+1}} - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}} s_i$$

where s_i is the transposition switching i and $i+1$ in S_n

<u>Proposition</u> Define $Y_1, \dots, Y_n$ by

$$Y_1 = g T_{n-1} \cdots T_1 \quad \text{and} \quad Y_{j+1} = T_j^{-1} Y_j T_j^{-1}.$$

Then, as operators on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n}$,

$$D_n^1 = Y_1 + \dots + Y_n \quad \text{and}$$

$$D_n^r = e_r(Y_1, \dots, Y_n) = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| = r}} Y_I,$$

where $Y_I = Y_{i_1} \cdots Y_{i_r}$ if $I = \{i_1, \dots, i_r\}$