Open boundary Hecke and Temperley-Lieb algebras

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MathPhys Seminar

4 May 2021

arXiv1804.10296 and arXiv2009.02812
with Zajj Daugherty

V. Rittenberg 1934–2018

$V = \mathbb{C}^2 = \mathbb{C}\text{span}\{v_\uparrow, v_\downarrow\} = \mathbb{C}\text{span}\{v_1, v_2\}$

spins

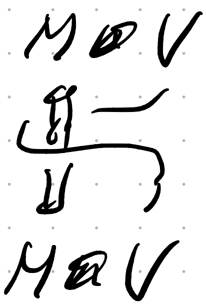
$V = \mathbb{C}^n = \mathbb{C}\text{-span}\{v_1, v_2, \ldots, v_n\}$.

$M \otimes \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ factors}} \otimes N$ spin chain with boundaries

$U = U_t\mathfrak{sl}_2$ and M, N, V are U-modules.

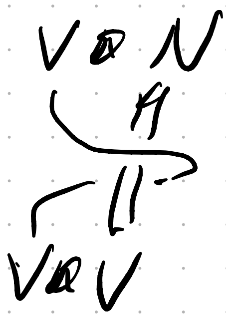
$U = U_t\mathfrak{gl}_n$

U has an R-matrix

$\check{R}_{MV}\check{R}_{VM} =$  $: M \otimes V \to M \otimes V$

$\check{R}_{VV} =$ 

U-module homomorphism (preserve

$\check{R}_{NV}\check{R}_{VN}$: 

: $V \otimes N \to V \otimes N$ (U-symmetry)

This gives a representation

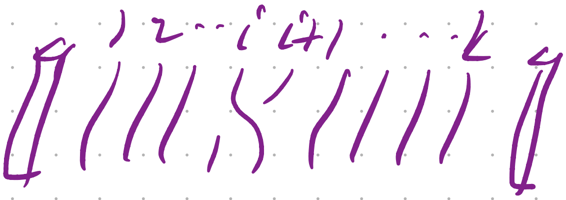
$$B_k \xrightarrow{\pi} \mathrm{End}_U(M \otimes V^{\otimes k} \otimes N)$$

The affine Hecke algebra H_k

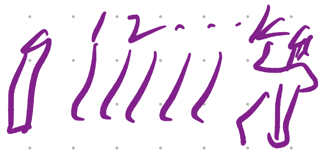
The **braid group** $\mathcal{B}_k$ for 

is generated by

$T_0 =$ 

$T_i =$ 

for $i \in \{1, \dots, k-1\}$

$T_k =$ 

with relations

$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$,

$T_{k-1} T_k T_{k-1} T_k = T_k T_{k-1} T_k T_{k-1}$,

$T_i T_{i+1} T_i = T_{i+1} T_i T_i$ and $T_j T_l = T_l T_j$

for $i \in \{1, \dots, k-2\}$ and $j, l \in \{0, 1, \dots, k\}$
with $l \notin \{j+1, j-1\}$

If there are $t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_k^{\frac{1}{2}} \in \mathbb{C}^\times$ with

$T_0 - T_0^{-1} = t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}$

$T_k - T_k^{-1} = t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}$ and $T_i^{\frac{1}{2}} - T_i^{-\frac{1}{2}} = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$

then π is a representation of the <u>affine Hecke algebra</u> H_k

$$H_k \xrightarrow{\pi} \mathrm{End}_{\mathcal{U}}(M \otimes V^{\otimes k} \otimes N)$$

$$T_0 \longmapsto \check{R}_{VM} \check{R}_{MV}$$

$$T_i \longmapsto \check{R}_i \quad \left(\begin{array}{l}\text{which is } \check{R}_{VV} \\ \text{in } i\text{th and } i+1\text{'st} \\ \text{factors of } V^{\otimes k}\end{array}\right)$$

$$T_k \longmapsto \check{R}_{NV} \check{R}_{VN}$$

How does π decompose into irreducibles?

What are properties of these irreducibles?

Irreducible H_k-representations

Kazhdan-Lusztig 1987 $t^{\frac{1}{2}} = t_0^{\frac{1}{2}} = t_k^{\frac{1}{2}}$

Syu Kato 2006: almost all $t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_k^{\frac{1}{2}}$

Theorem There is a bijection

$$\left\{ \begin{array}{l} \text{irreducible} \\ H_k \text{ representations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{pairs } (\gamma, J) \text{ with} \\ \gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{C}^k / W_0 \\ J \subseteq \left\{ \begin{array}{l} H^{\xi_i}, H^{\xi_i + r_1}, H^{\xi_i + r_2}, \\ H^{\xi_i - \xi_j}, H^{\xi_i - \xi_j + 1} \end{array} \right\} \\ \text{such that} \dots \end{array} \right\}$$

$$H^{(\gamma, J)} \longleftarrow\!\shortmid (\gamma, J)$$

where

$$W_0 = D_k(\mathbb{Z}) = \{ A \in M_k(\mathbb{Z}) \mid AA^t = 1 \}$$

$= \left\{ \begin{array}{l} A \in M_k(\mathbb{Z}) \text{ such that} \\ \text{(a) exactly one nonzero entry} \\ \text{in each row and each column} \\ \text{(b) nonzero entries are } \pm 1 \end{array} \right\}$

Kazhdan-Lusztig and Kato used techniques:

(A) Murphy elements

(B) $Z(H_k)$

(C) Geometry of Springer Fibres

(A) and (B) were developed in the q-adic groups literature in the 1970's: Rodier, Casselman, Bernstein, Zelevinsky ... LUSZTIG following Harish-Chandra.

Murphy elements

$y_i =$  for $i \in \{1, \dots, k\}$.

$y_1, y_2, \dots, y_k$ commute with each other.

The centre $Z(H_k)$

$$Z(H_k) = \mathbb{C}[y_1^{\pm 1}, \dots, y_k^{\pm 1}]^{W_0}$$

$$= \left\{ f \in \mathbb{C}[y_1^{\pm 1}, \dots, y_k^{\pm 1}] \;\middle|\; \begin{array}{l} f(y_1, \dots, y_i, \dots, y_k) \\ \quad = f(y_1, \dots, y_i^{-1}, \dots, y_k) \\ f(y_1, \dots, y_i, y_{i+1}, \dots, y_k) \\ \quad = f(y_1, \dots, y_{i+1}, y_i, \dots, y_k) \end{array} \right\}$$

If $H^{(\lambda, J)}$ is an irreducible H_k-representation then

$f(y_1, \dots, y_k)$ acts on $H^{(\lambda, J)}$

by $f(e^{\gamma_1}, \dots, e^{\gamma_k}) = Id.$

Temperley-Lieb for 

Let $p_i^{(1^3)}$, $\quad p_0^{(1^2,\emptyset)}$, $p_0^{(\emptyset,1^2)}$,

$p_k^{(1^2,\emptyset)}$, $p_k^{(\emptyset,1^2)}$

be given by ....

The <u>Temperley-Lieb algebra</u> TL_k is H_k with the additional relations

$$p_i^{(1^3)}=0,\quad p_0^{(1^2,\emptyset)}=p_0^{(\emptyset,1^2)},\quad p_k^{(\emptyset,1^2)}=p_k^{(1^2,\emptyset)}$$

<u>Conversion to diagrams</u> Let

$a=\pm 1$, $\quad a_0 a = t^{-\frac{1}{2}}t_0^{\frac{1}{2}}+t^{\frac{1}{2}}t_0^{-\frac{1}{2}}$

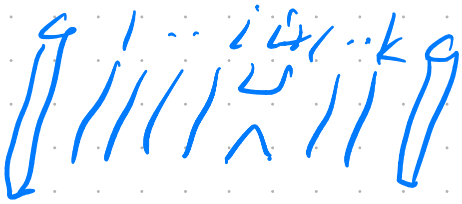
$$a_k a = t^{-\frac{1}{2}}t_k^{\frac{1}{2}}+t^{\frac{1}{2}}t_k^{-\frac{1}{2}}$$

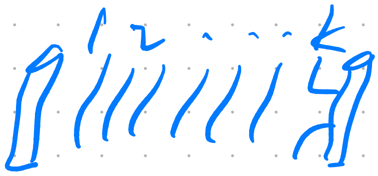
and define e_0, e_k and e_i by

$T_0 = a_0 e_0 + t_0^{\frac{1}{2}}$

$T_k = a_k e_k + t_k^{\frac{1}{2}}$ and $T_i = a e_i + t^{\frac{1}{2}}$.

Use

$e_0 =$  $e_i =$ 

$e_k =$ 

and rewrite all products in TL_k in terms of diagrams.

$H_k \twoheadrightarrow TL_k$ gives $\mathrm{Rep}(TL_k) \subseteq \mathrm{Rep}(H_k)$

$\mathrm{Rep}(TL_k)$ and structure of most $H^{(\gamma, J)}$ determined by DeGier-Nichols 0703338.

Daugherty-R. 1804.10296 & 2009.02812

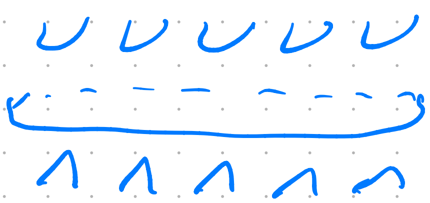
(1) We determine exactly when $H^{\langle \gamma, J\rangle}$ has a basis of simultaneous eigenvectors for $Y_1, \dots, Y_k$

(2) For each of the $H^{\langle \gamma, J\rangle}$ in (1) we give explicit formulas for the action of T_i on the Gelfand-Tsetlin basis (indexed by 180° rotationally symmetric tableaux).

(3) We determine exactly which $H^{\langle \gamma, J\rangle}$ are in $\mathrm{Rep}(TL_k)$.

(4) $Z(TL_k) = \mathbb{C}[z]$ where

$$z = \left(\frac{t^{k} - t^{-k}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}\right)$$



$$= Y_1 + Y_1^{-1} + \cdots + Y_k + Y_k^{-1}$$

(5) We determine exactly which $\mathcal{H}^{(\delta,J)}$ are components of a two boundary spin chain $M \otimes V^{\otimes k} \otimes N$ for $U_q\mathfrak{gl}_n$ and $U_q\mathfrak{sl}_2$.

$P_i^{(1^3)}$ is the combination of T_i, T_{i+1} such that

$$T_i P_i^{(1^3)} = -t^{-\frac{1}{2}} P_i^{(1^3)} \text{ and } T_{i+1} P_i^{(1^3)} = -t^{-\frac{1}{2}} P_i^{(1^3)}$$

alternatively

$$P_i^{(1^3)} = T_i T_{i+1} T_i - t^{\frac{1}{2}} T_i T_{i+1} - t^{\frac{1}{2}} T_{i+1} T_i + t T_i + t T_{i+1} - t^{3/2}$$

$P_k^{(1^2,\emptyset)}$ and $P_k^{(\emptyset,1^2)}$ are the combinations of T_{k-1}, T_k such that

$$T_k P_k^{(1^2,\emptyset)} = t_k^{\frac{1}{2}} P_k^{(1^2,\emptyset)}, \quad T_{k-1} P_k^{(1^2,\emptyset)} = -t^{-\frac{1}{2}} P_k^{(1^2,\emptyset)}$$

$$T_k P_k^{(\emptyset,1^2)} = -t_k^{-\frac{1}{2}} P_k^{(\emptyset,1^2)}, \quad T_{k-1} P_k^{(\emptyset,1^2)} = -t^{-\frac{1}{2}} P_k^{(\emptyset,1^2)}$$

alternatively,

$$P_k^{(1^2,\emptyset)} = T_k T_{k-1} T_k T_{k-1} + t_k^{-\frac{1}{2}} T_{k-1} T_k T_{k-1} - t^{\frac{1}{2}} T_k T_{k-1} T_k - t_k^{-\frac{1}{2}} t^{\frac{1}{2}} T_k T_{k-1} - t_k^{-\frac{1}{2}} t^{\frac{1}{2}} T_{k-1} T_k - t_k t^{\frac{1}{2}} T_{k-1} + t_k^{-\frac{1}{2}} t T_k + t_k t$$

and

$$p_k^{(\Phi,1^2)} = T_k T_{k-1} T_k T_{k-1} - t_k^{\frac12} T_{k-1} T_k T_{k-1} - t^{\frac12} T_k T_{k-1} T_k + t_k^{\frac12} t^{\frac12} T_k T_{k-1} + t_k^{\frac12} t^{\frac12} T_{k-1} T_k - t_k t^{\frac12} T_{k-1} - t_k^{\frac12} t T_k + t_k t$$

$p_0^{(1^2,\Phi)}$ and $p_0^{(\Phi,1^2)}$ are the combinations of T_0 and T_1 such that

$$T_0 p_0^{(1^2,\Phi)} = t_0^{\frac12} p_0^{(1^2,\Phi)}, \quad T_1 p_0^{(1^2,\Phi)} = -t^{-\frac12} p_0^{(1^2,\Phi)}$$

$$T_0 p_0^{(\Phi,1^2)} = -t_0^{-\frac12} p_0^{(\Phi,1^2)}, \quad T_1 p_0^{(\Phi,1^2)} = -t^{-\frac12} p_0^{(\Phi,1^2)}$$

alternatively

$$p_0^{(1^2,\Phi)} = T_0 T_1 T_0 T_1 + t_0^{-\frac12} T_1 T_0 T_1 - t^{\frac12} T_0 T_1 T_0 - t_0^{-\frac12} t^{\frac12} T_0 T_1 - t_0^{-\frac12} t^{\frac12} T_1 T_0 - t_0 t^{\frac12} T_1 + t_0^{-\frac12} t T_0 + t_0 t,$$

and

$$p_0^{(\Phi,1^2)} = T_0 T_1 T_0 T_1 - t_0^{\frac12} T_1 T_0 T_1 - t^{\frac12} T_0 T_1 T_0 + t_0^{\frac12} t^{\frac12} T_0 T_1 + t_0^{\frac12} t^{\frac12} T_1 T_0 - t_0 t^{\frac12} T_1$$

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