Representation Theory seminar — 08.05.2021

Kac Moody algebra $\mathfrak{g}$ — 11 May Talk ①

Generators: $e_0, e_1, \ldots, e_n$
$f_0, f_1, \ldots, f_n$ and $d_1, \ldots, d_\ell$
$h_0, h_1, \ldots, h_n$

Relations: Serre relations

$\mathfrak{h}$ has basis $\{h_0, h_1, \ldots, h_n, d_1, \ldots, d_\ell\}$

$\mathfrak{n}^+$ is subalgebra generated by $e_0, e_1, \ldots, e_n$

$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$

Fundamental weights

$\mathfrak{h}^*$ has basis $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \delta_1, \delta_2, \ldots, \delta_\ell\}$

given by

$\Lambda_i(h_j) = \delta_{ij}, \qquad \Lambda_i(d_j) = 0$

$\Lambda_i(h_j) = 0 \qquad \delta_i(d_j) = \delta_{ij}.$

Roots As an $\mathfrak{h}$-module

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \quad \text{with } \mathfrak{g}_0 = \mathfrak{h}$$

Define $\alpha_0, \alpha_1, \ldots, \alpha_n$ by

$e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i = \mathfrak{g}_{-\alpha_i}$ for $i \in \{0, 1, \ldots, n\}$.

Weyl group

$W \subseteq GL(\mathfrak{h}^*)$ generated by

$s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ given by $s_i\lambda = \lambda - \lambda(h_i)\alpha_i$

W acts on $\mathfrak{h}$ by $s_i h = h - \alpha_i(h) h_i$.

For $\beta^\vee \in W \cdot \{h_0, h_1, \dots h_n\}$ define

$$s_{\beta^\vee} = w s_i w^{-1} \quad \text{if} \quad \beta^\vee = w h_i.$$

The element ρ

$\rho = \Lambda_0 + \Lambda_1 + \dots + \Lambda_n$ so that $\rho(h_i) = 1$

for $i \in \{0, 1, \dots, n\}$. The <u>dot</u> <u>action</u> of W on $\mathfrak{h}^*$ (or ρ-shifted action) is

$$w \circ \lambda = w(\lambda + \rho) - \rho.$$

Vermamodules For $\lambda \in \mathfrak{h}^*$

$$M(\lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda = U\mathfrak{n}^- v_\lambda$$

where

$\mathfrak{n}^+$ is the subalgebra gen. by $e_0, e_1, \ldots, e_n$

$\mathfrak{n}^-$ is the subalgebra gen. by $f_0, f_1, \ldots, f_n$

$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$

$M(\lambda)$ has a unique simple quotient $L(\lambda)$.

Linkage Let

$W(\lambda) \subseteq W$ be the subgroup gen. by

$s_{\beta^\vee}$ for $\beta^\vee \in W \cdot \{h_0, h_1, \ldots, h_n\}$ with $\lambda(\beta^\vee) \in \mathbb{Z}$.

(a) Usually $W(\lambda) = \{1\}$.

(b) λ is integral if $W(\lambda) = W$.

Theorem If $L(\mu)$ is a composition factor of $M(\lambda)$ then

$$\mu \in W(\lambda) \circ \lambda.$$

Kazhdan-Lusztig basis

Let $\mathfrak{h}^*$ be a set of representatives of the $W(\lambda)\circ$ orbits on $\mathfrak{h}^*$.

For $\nu \in \hat{\mathfrak{h}}^*$ let W^ν be a set of representatives of

$$W(\nu) \big/ \mathrm{Stab}^\circ(\nu)$$

Let K^ν be the free $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module with basis $\{A_w \mid w \in W^\nu\}$.

Partial order on W^ν

$w \geq v$ if $\quad w\circ\nu - v\circ\nu \in \mathbb{Z}_{\geq 0}\text{-span}\{\alpha_0, \dots, \alpha_n\}$

Bar involution $\quad \bar{\ } : K^\nu \to K^\nu$

$\mathbb{Z}$-linear, $\quad \overline{t^{\frac{1}{2}} m} = t^{-\frac{1}{2}} \overline{m}, \quad \overline{\overline{m}} = m$

$$\overline{A_w} = A_w + \sum_{v<w} a_{vw} A_v$$

for $m \in M$, $w \in W^\nu$. For $w \in W^\nu$ let C_w be defined by

$\overline{C_w} = C_w$, $\quad C_w = A_w + \sum_{v<w} p_{vw} A_v$ with $p_{vw} \in t^{\frac{1}{2}}\mathbb{Z}[t^{\frac{1}{2}}]$.

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Kazhdan-Lusztig idea

Let $F^{\cdot}(M)$ be the radical / socle / Jantzen filtration of M (with semisimple layers). Then

$$P_{vw}(t) = \sum_i \left[L(v \circ v) : \frac{F^i(M(w \circ v))}{F^{i+1}(M(w \circ v))} \right] t^i$$

Basic heuristics

(a) C_w exist and are well defined if the intervals in $(W^v, \leq)$ are finite

(b) P_{vw} depends only on the interval $[v, w]$ in the poset $(W^v, \leq)$.

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ν in Negative Tits cone

Let $\nu \in \hat{\mathfrak{h}}^*$ with

$(\nu+\rho)(h_{\beta^\vee}) \in \mathbb{Z}_{\leq 0}$ for $\beta^\vee$ with $s_{\beta^\vee} \in W(\nu)$.

$H(\nu)$ is the $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ algebra with

(a) basis $\{T_w \mid w \in W(\nu)\}$

(b) $T_v T_w = T_{vw}$ if $\ell(vw) = \ell(v) + \ell(w)$

(c) $T_{s_i}^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_i + 1$.

Let $K^\nu = H \mathbb{1}_\nu$ with $T_{s_i} \mathbb{1}_\nu = t^{\frac{1}{2}} \mathbb{1}_\nu$
if $s_i \in \mathrm{Stab}^0(\nu)$.

Then $K^\nu = H \mathbb{1}_\nu$ has basis

$\{A_w \mid w \in W^\nu\}$ where $A_w = T_w \mathbb{1}_\nu$ for $w \in W^\nu$.

<u>Partial order</u> $v \leq w$ is the subword order
($w = s_{i_1} \cdots s_{i_\ell}$ as a product of simple refl.).

<u>Bar involution</u> $\overline{T_i} = T_i^{-1}$, $\overline{t^{\frac{1}{2}}} = t^{-\frac{1}{2}}$,

$\overline{hm} = \overline{h}\,\overline{m}$, $\overline{A_w} = \overline{T_w} \mathbb{1}_\nu$,

for $h \in H(\nu)$ and $m \in K^\nu = H(\nu) \mathbb{1}_\nu$.

Kazhdan-Lusztig basis

$$\overline{C_w} = C_w \quad \text{and} \quad C_w = A_w + \sum_{v \le w} P_{vw} A_v$$

with $P_{vw} \in t\mathbb{Z}[t]$.

$M(w \circ v)$ has a finite composition series

$$[M(w \circ v)] = \sum_{v \le w} P_{vw}(1)\, [L(v \circ v)]$$

$-v$ in positive Tits cone

Use the same $K^v = H(v) \mathbb{1}_v$

with the same bar involution

and the reverse partial order on $W^v = W^{-v}$.

Let

$$\overline{C^w} = C^w \quad \text{and} \quad C^w = A_w + \sum_{v \ge w} Q_{vw} A_v$$

with $Q_{vw} \in t^{-1}\mathbb{Z}[t^{-1}]$. Then

$$\begin{array}{c} \\ 1 \\ s_1 \\ \vdots \end{array} \begin{array}{c} 1 \quad s_1 \quad \cdots \\ \left(\begin{array}{ccc} & & \\ & Q_{wv} & \\ & & \end{array} \right) \end{array} \begin{array}{c} 1 \quad s_1 \quad \cdots \\ \left(\begin{array}{ccc} & & \\ & P_{vw} & \\ & & \end{array} \right) \end{array} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$M(-w \circ v)$ has an infinite composition series

$$[M(-w \circ v)] = \sum_{v \ge w} Q_{vw}(1)\, [L(-v \circ v)].$$