Representation Theory seminar
The affine Weyl group

21.05.2021
25 May Talk 1
A. Ram

$P^\vee$ has $\mathbb{Z}$-basis $\{\omega_1^\vee, \ldots, \omega_n^\vee\}$
$\cup\!|$
$Q^\vee$ has $\mathbb{Z}$-basis $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$

W_0 is a finite group acting on $P^\vee$ (and $Q^\vee$) generated by $s_1, \ldots, s_n$ where

$s_i : P^\vee \to P^\vee$ is given by $s_i\lambda^\vee = \lambda^\vee - \lambda^\vee(\alpha_i)\alpha_i^\vee$.

where $\lambda^\vee(\alpha_i) = \lambda_i$ if $\lambda^\vee = \lambda_1\omega_1^\vee + \cdots + \lambda_n\omega_n^\vee$.

The <u>affine Weyl group</u> is

$$W = \{t_{\lambda^\vee} w \mid \lambda^\vee \in P^\vee, w \in W_0\} \text{ with}$$

$$(t_{\lambda^\vee} u)(t_{\mu^\vee} v) = t_{\lambda^\vee} t_{u\mu^\vee} uv = t_{\lambda^\vee + u\mu^\vee} uv$$

for $\lambda^\vee, \mu^\vee \in P^\vee$ and $u, v \in W_0$.

$$W_{Q^\vee} = \{t_{\lambda^\vee} w \mid \lambda^\vee \in Q^\vee, w \in W_0\}$$

is a subgroup of W. Let

$$s_0 = t_{\theta^\vee} s_\theta$$

Then $W_{Q^\vee}$ is presented by generators $s_0, s_1, \ldots, s_n$ with

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$$

where m_{ij} is the order of $s_i s_j$ in W.

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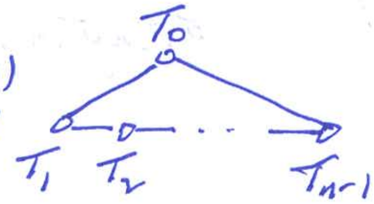
(1.B)

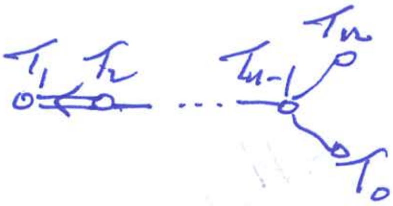
Affine Dynkin diagrams

$A_{n-1}^{(1)}$



$\Omega \cong \mathbb{Z}/n\mathbb{Z}$

$B_n^{(1)}$

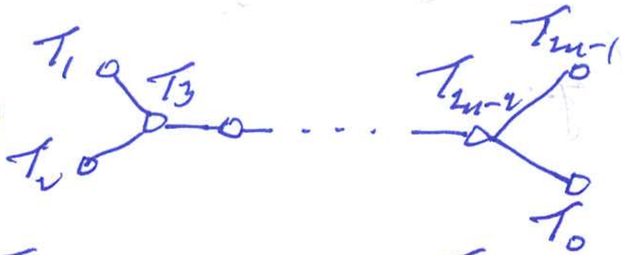


$\Omega \cong \mathbb{Z}/2\mathbb{Z}$

$C_n^{(1)}$

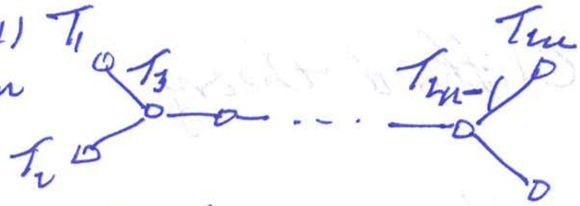
$\Omega \cong \mathbb{Z}/2\mathbb{Z}$

$D_{2n-1}^{(1)}$



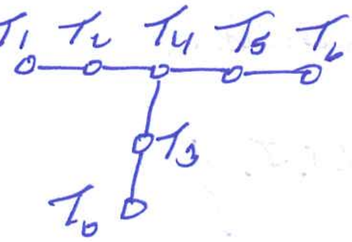
$\Omega \cong \mathbb{Z}/4\mathbb{Z}$

$D_{2n}^{(1)}$



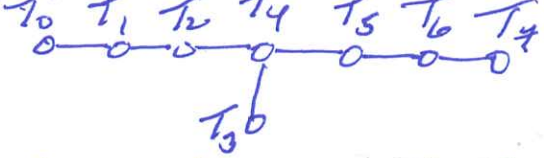
$\Omega \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$E_6^{(1)}$



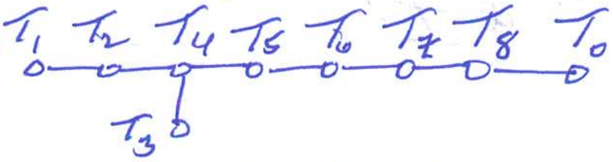
$\Omega \cong \mathbb{Z}/3\mathbb{Z}$

$E_7^{(1)}$



$\Omega \cong \mathbb{Z}/2\mathbb{Z}$

$E_8^{(1)}$



$\Omega \cong \{1\}$

$F_4^{(1)}$

$\Omega \cong \{1\}$

$G_2^{(1)}$

$\Omega \cong \{1\}.$

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The affine Hecke algebra H

H is the $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-algebra with basis

$$\{y^{\lambda^\vee} T_w \mid \lambda^\vee \in P^\vee, w \in W_0\} \quad \text{with}$$

$$\underbrace{T_{s_i} T_{s_j} \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_{s_j} T_{s_i} \cdots}_{m_{ij} \text{ factors}}, \qquad y^{\lambda^\vee} y^{\mu^\vee} = y^{\lambda^\vee + \mu^\vee} = y^{\mu^\vee} y^{\lambda^\vee},$$

$$T_{s_i}^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_{s_i} + 1,$$

$$T_{s_i} y^{\lambda^\vee} = y^{s_i \lambda^\vee} T_{s_i} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{y^{\lambda^\vee} - y^{s_i \lambda^\vee}}{1 - y^{-\alpha_i^\vee}}$$

for $i, j \in \{1, \ldots, n\}$ and $\lambda^\vee, \mu^\vee \in P^\vee$.

Then

$$H_{Q^\vee} = \mathrm{span}\{y^{\lambda^\vee} T_w \mid \lambda^\vee \in Q^\vee, w \in W_0\}$$

is a subalgebra of H. Let

$$T_{s_0} = y^{\theta^\vee} T_{s_\theta}$$

Then $H_{Q^\vee}$ is presented by generators $T_{s_0}, T_{s_1}, \ldots, T_{s_n}$ with

$$T_{s_i}^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_{s_i} + 1 \quad \text{and}$$

$$\underbrace{T_{s_i} T_{s_j} T_{s_i} \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_{s_j} T_{s_i} T_{s_j} \cdots}_{m_{ij} \text{ factors}}$$

for $i, j \in \{0, 1, \ldots, n\}$.

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Isogeny

Let

$$\Omega = P^\vee / Q^\vee = \{ g \in W \mid \ell(g) = 0 \}$$

$$= \mathrm{Aut}(\text{affine Dynkin diagram}).$$

An element $g_i \in \Omega$ is determined by i where

$$g_i(0) = i \quad \left(\begin{array}{l}\text{action on vertices of affine} \\ \text{Dynkin diagram}\end{array}\right).$$

Let $W_{\{i\}} = \langle s_1, \ldots, \hat{s}_i, \ldots, s_n \rangle \subseteq W_0$,

w_i the longest element of $W_{\{i\}}$,

w_0 the longest element of W_0.

Proposition (a) There is an injective group homomorphism

$$\begin{aligned} \Omega &\hookrightarrow H^\times \\ g_i &\longmapsto y^{\omega_i^\vee} T_{w_0 w_i} \end{aligned}$$

(b) $H = \Omega \ltimes H_{Q^\vee}$ with

$$\gamma T_{s_j} \gamma^{-1} = T_{s_{\gamma(j)}} \quad \text{for } \gamma \in \Omega \text{ and } j \in \{0, 1, \ldots, n\}$$

Comparing representations of H and $H_{Q^\vee}$

(see R-Ramagge arXiv:0401322
and Reeder Rep. Theory 6 (2002) 101-126)

In all cases except $D_{2n}^{(1)}$ the group

Ω is cyclic.

Define a homomorphism $\begin{array}{ccc}\Omega & \to & \mathrm{Aut}(H) \\ g & \mapsto & \varphi_g\end{array}$ by

$$\begin{array}{rcl}\varphi_g : H & \longrightarrow & H \\ w & \longmapsto & e^{2\pi i/\ell} w \\ T_{s_i} & \longmapsto & T_{s_i}\end{array}$$

where w generates Ω

for $i \in \{0,1,\ldots,n\}$

and ℓ is the order of g.

So Ω acts on H.

Theorem $H_{Q^\vee} = H^\Omega$

By applying a version of Clifford theory (see Appendix in arXiv:0401322) we get

Corollary Let $H^{(\gamma,J)}$ be a simple H-module

Let

$$\mathcal{I} = \{ g \in \Omega \mid \varphi_g^*(H^{(\gamma,J)}) \cong H^{(\gamma,J)} \}.$$

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As an $H_{Q^\vee} \times \mathcal{I}$ bimodule

$$H^{(\gamma,J)} \cong \bigoplus_{\rho\in\hat{\mathcal{I}}} H^{(\gamma,J,\rho)} \otimes \mathcal{I}^\rho$$

where
$\hat{\mathcal{I}}$ is an index set for simple $\mathcal{I}$-modules

$\mathcal{I}^\rho$ is the simple $\mathcal{I}$-module indexed by ρ.

The nonzero
$H^{(\gamma,J,\rho)}$ are a complete set of
nonisomorphic $H_{Q^\vee}$-modules.

Finite Hecke algebras for A_n, B_n, D_n

Let $r, p, n \in \mathbb{Z}_{>0}$ and assume p divides r.

$$G_{r,p,n} = \left\{ w \in M_n(\mathbb{C}) \;\middle|\; \begin{array}{l} \text{(a) exactly one nonzero entry in} \\ \quad \text{each row and each column} \\ \text{(b) nonzero entries are } r^{\text{th}} \text{ roots of } 1 \\ \text{(c) } \Big(\prod_{w_{ij} \neq 0} w_{ij}\Big)^{r/p} = 1. \end{array} \right\}$$

Then

$$G_{1,1,n} = S_n = W_0 \text{ for } A_{n-1}$$

$$G_{2,1,n} = W_0 \text{ for } B_n \text{ and } C_n$$

$$G_{2,2,n} = W_0 \text{ for } D_n$$

Let $u_1, \ldots, u_r \in \mathbb{C}$.

The <u>Hecke algebra of $G_{r,1,n}$</u> (see Ariki-Koike, Broué-Malle-Rouquier)

is $H_{r,1,n}(u_1, \ldots, u_r; t^{\frac12})$ generated by

$y^{\varepsilon_1^\vee}, T_1, \ldots, T_{n-1}$ with relations

$y^{\varepsilon_1^\vee}$ o═o T_1 — o T_2 … — o T_{n-1}

$$(T_i - t^{\frac12})(T_i + t^{-\frac12}) = 0$$

$$(y^{\varepsilon_1^\vee} - u_1) \cdots (y^{\varepsilon_1^\vee} - u_r) = 0.$$

The <u>Hecke algebra of $G_{r,p,n}$</u> is $H_{r,p,n}(u_1,\ldots,u_r; t^{1/2})$, the subalgebra of
$$H_{r,1,n}(u_1,\ldots,u_r; t^{1/2})$$
generated by

$$a_0 = (y^{\varepsilon_1^\vee})^p, \quad a_1 = y^{-\varepsilon_1^\vee} T_1 y^{\varepsilon_1^\vee} \quad \text{and}$$
$$a_{i+1} = T_i \quad \text{for } i \in \{1,\ldots,n-1\}.$$

The <u>affine Hecke algebra of type GL_n</u> is $H_{GL_n} = H_{\infty,1,n}$ generated by $y^{\varepsilon_1^\vee}, T_1, \ldots, T_{n-1}$ with

$$\overset{y^{\varepsilon_1^\vee}}{\circ} = \overset{T_1}{\circ} - \overset{T_2}{\circ} - \cdots - \overset{T_{n-1}}{\circ} \quad \text{and}$$

$$(T_i - t^{1/2})(T_i + t^{-1/2}) = 0.$$

Here we use Coxeter diagram shorthand for relations

$\overset{a}{\circ} \quad \overset{b}{\circ}$ indicates $ab = ba$

$\overset{a}{\circ} - \overset{b}{\circ}$ indicates $aba = bab$

$\overset{a}{\circ} = \overset{b}{\circ}$ indicates $abab = baba$

$\overset{a}{\circ} \equiv \overset{b}{\circ}$ indicates $ababab = bababa$

Theorem

(a) There is a surjective homomorphism

$$\begin{aligned} H_{GL_n} &\twoheadrightarrow H_{r,1,n} \\ y^{\varepsilon_i^\vee} &\longmapsto y^{\varepsilon_i^\vee} \\ T_i &\longmapsto T_i \quad \text{for } i \in \{1,\ldots,n-1\} \end{aligned}$$

(b) Define an action of $\mathbb{Z}/p\mathbb{Z} = \{1, g, \ldots, g^{p-1}\}$ on $H_{r,1,n}$ (by algebra automorphisms)

$$\begin{aligned} \varphi_g : H_{r,1,n} &\longrightarrow H_{r,1,n} \\ T_i &\longmapsto T_i \quad \text{for } i \in \{1,\ldots,n-1\} \\ y^{\varepsilon_i^\vee} &\longmapsto e^{2\pi i/p} y^{\varepsilon_i^\vee} \end{aligned}$$

Then $H_{r,p,n} = (H_{r,1,n})^{\mathbb{Z}/p\mathbb{Z}}$

So all irreducible representations of $H_{r,p,n}$ are obtained from irreducible representations of H_{GL_n} and a little bit of (easy) Clifford theory.