Representations of affine Hecke algebras 31.05.2021 ①
Central characters, Weight Spaces, Intertwiners Rep Thy Seminar
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Affine Hecke algebra H.

H has basis $\{T_w Y^{\lambda^\vee} \mid \lambda^\vee \in P^\vee, w \in W_0\}$

with $$T_{s_i} Y^{\lambda^\vee} = Y^{s_i\lambda^\vee} T_{s_i} + \frac{t^{\frac12}-t^{-\frac12}}{1-Y^{-\alpha_i^\vee}}\left(Y^{\lambda^\vee}-Y^{s_i\lambda^\vee}\right)$$

$P^\vee$ has $\mathbb{Z}$-basis $\{\omega_1^\vee, \ldots, \omega_n^\vee\}$

W_0 is a finite group acting on $P^\vee$ generated by $s_1, \ldots, s_n$ where
$s_i\lambda^\vee = \lambda^\vee - \lambda_i\alpha_i^\vee$ where $\lambda^\vee = \lambda_1\omega_1^\vee + \cdots + \lambda_n\omega_n^\vee$.

Subalgebras

$\mathbb{C}[Y] = \mathrm{span}\{Y^{\lambda^\vee} \mid \lambda^\vee \in P^\vee\} = \mathbb{C}[Y^{\pm\omega_1^\vee}, \ldots, Y^{\pm\omega_n^\vee}]$

$$Y^{\lambda^\vee}Y^{\mu^\vee} = Y^{\lambda^\vee+\mu^\vee} = Y^{\mu^\vee}Y^{\lambda^\vee}$$

$H_{fin} = \mathrm{span}\{T_w \mid w \in W_0\}$

$T_{s_i}^2 = (t^{\frac12}-t^{-\frac12})T_{s_i} + 1$,

$T_w = T_{s_{i_1}}\cdots T_{s_{i_\ell}}$ if $w = s_{i_1}\cdots s_{i_\ell}$ is reduced.

$Z(H) = \mathbb{C}[Y]^{W_0} = \left\{ f \in \mathbb{C}[Y] \;\middle|\; \begin{matrix}\text{If } w \in W_0 \text{ then} \\ wf = f\end{matrix} \right\}$

where $wY^{\lambda^\vee} = Y^{w\lambda^\vee}$

<u>The torus $T^\vee$ (irred. reps of $\mathbb{C}[Y]$)</u>

$$T^\vee = \mathrm{Hom}(P^\vee, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^n$$

If $\gamma \in T^\vee$ then

$$\begin{aligned} \gamma : \mathbb{C}[Y] &\to \mathbb{C} \\ y^{\omega_i^\vee} &\mapsto \gamma_i \end{aligned} \quad \text{with } (\gamma_1, \ldots, \gamma_n) \in (\mathbb{C}^\times)^n$$

W_0 acts on $T^\vee$ by

$$(w\gamma)(y^{\lambda^\vee}) = \gamma(y^{w^{-1}\lambda^\vee})$$

<u>Central characters</u>

Let M be a fin. dim'l simple H-module.

If $f \in Z(H) = \mathbb{C}[Y]^{W_0}$ then

f acts on M by a constant.

But $f \in \mathbb{C}[y^{\pm\omega_1^\vee}, \ldots, y^{\pm\omega_n^\vee}]$. There exists $\gamma \in T^\vee$ such that if $m \in M$ and $f \in Z(H)$ then

$$fm = \gamma(f)m.$$

(evaluate $y^{\omega_i^\vee}$ to the value γ_i).

The <u>central character of M</u> is

the orbit $W_0\gamma$.

Weight spaces

Let M be a finite dimensional H-module.

Let $\gamma \in T^\vee$.

$$M_\gamma^{gen} = \left\{ m \in M \;\middle|\; \begin{array}{l}\text{There exists } k \in \mathbb{Z}_{>0} \text{ such that} \\ (y^{\lambda^\vee} - \gamma(y^{\lambda^\vee}))m = 0, \text{ for } \lambda^\vee \in P^\vee \end{array} \right\}$$

$$\cup\!|$$

$$\begin{aligned} M_\gamma &= \{ m \in M \mid y^{\omega_i} m = \gamma_i m \text{ for } i \in \{1, \ldots, n\} \} \\ &= \{ m \in M \mid (y^{\lambda^\vee} - \gamma(y^{\lambda^\vee}))m = 0 \}. \end{aligned}$$

M_γ is the <u>γ-weight space of M</u>,

M_γ^{gen} is the <u>generalised γ-weight space of M</u>.

<u>Interwiners</u> For $i \in \{1, \ldots, n\}$ define

$$\tau_i^\vee = T_i + \frac{t^{\frac12} - t^{-\frac12}}{1 - y^{-\alpha_i^\vee}}$$

then

$$\tau_i^\vee y^{\lambda^\vee} = y^{s_i \lambda^\vee} \tau_i^\vee,$$

$$\underbrace{\tau_i^\vee \tau_j^\vee \tau_i^\vee \cdots}_{m_{ij} \text{ factors}} = \underbrace{\tau_j^\vee \tau_i^\vee \tau_j^\vee \cdots}_{m_{ij} \text{ factors}}$$

$$(\tau_i^\vee)^2 = \frac{(t^{-\frac12} - t^{\frac12} y^{\alpha_i^\vee})(t^{-\frac12} - t^{\frac12} y^{-\alpha_i^\vee})}{(1 - y^{\alpha_i^\vee})(1 - y^{-\alpha_i^\vee})}$$

$\tau_w^\vee = \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee$ if $w = s_{i_1} \cdots s_{i_\ell}$ is reduced.

Intertwiners continued

Let M be a fin. dim'l H-module.

Let $\gamma \in T^\vee$ and assume $\gamma(y^{\alpha_i^\vee}) \neq 1$.

$$\begin{aligned} \tau_i^\vee : M_\gamma^{gen} &\longrightarrow M_{s_i\gamma}^{gen} \\ m &\longmapsto \tau_i^\vee m \end{aligned}$$ is defined and

$\tau_i^\vee$ is invertible if $\gamma(y^{\alpha_i^\vee}) \notin \{t, t^{-1}\}$. Hence,

if $\gamma\left((t^{-\frac{1}{2}} - t^{\frac{1}{2}} y^{\alpha_i^\vee})(t^{-\frac{1}{2}} - t^{\frac{1}{2}} y^{-\alpha_i^\vee})\right) \neq 0$.

then $\dim(M_\gamma^{gen}) = \dim(M_{s_i\gamma}^{gen})$.

<u>Formal characters</u> Assume M has central character $W_0 s$. Then

$$M = \bigoplus_{\gamma \in W_0 s} M_\gamma^{gen} = \bigoplus_{w \in W^s} M_{ws}^{gen}$$

where
$$\begin{aligned} W^s &= \{\text{min. length reps of } W_0/\mathrm{Stab}(s)\} \\ &= \{w \in W \mid \mathrm{Inv}(w) \cap Z(s) = \phi\} \end{aligned}$$

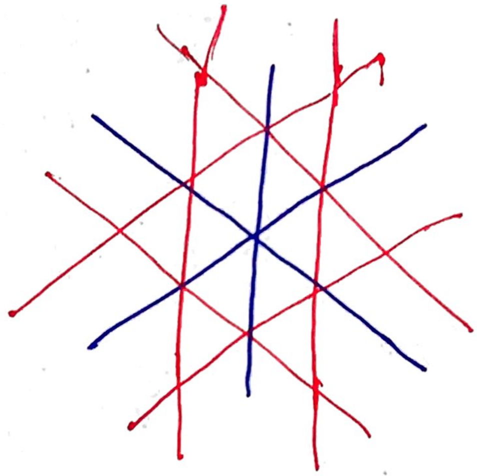
where $\mathrm{Inv}(w) = \{\beta^\vee \in (R^\vee)^+ \mid w\beta^\vee \notin (R^\vee)^+\}$

The <u>formal character of M</u> is

$$\sum_{w \in W^s} \dim(M_{ws}^{gen}) e^{ws}$$

(in $\mathbb{C}$-span$\{e^\gamma \mid \gamma \in T^\vee\}$.

<u>Local regions (s,J)</u> Let $s \in T^\vee$.



$$Z(s) = \{\beta \in R^\vee \mid s(Y^{\beta^\vee}) = 1\}$$

$= \{$blue lines s is on$\}$

$$P(s) = \{\beta \in R^\vee \mid s(Y^{\beta^\vee}) \in \{t, t^{-1}\}\}$$

$= \{$red lines s is on$\}$

<u>Idea</u> There is a bijection (not true without more adjectives)

$$\{\text{simple } H\text{-modules}\} \longleftrightarrow \left\{\begin{array}{l}\text{pairs } (s,J) \text{ with} \\ J \subseteq P(s)\end{array}\right\} \Big/ W_0$$

$$H^{(s,J)} \longleftarrow\!\shortmid \ (s,J)$$

Define $$F^{(s,J)} = \left\{ w \in W_0 \ \middle|\ \begin{array}{l} \mathrm{Inv}(w) \cap Z(s) = \emptyset \\ \mathrm{Inv}(w) \cap P(s) = J \end{array} \right\}$$

so that $$W^s = \bigcup_J F^{(s,J)}$$

A <u>local region</u> is a pair (s,J) with $F^{(s,J)} \neq \emptyset$.

Let M be a simple H-module with central char. s.

$$M = \bigoplus_{\substack{\text{local regions} \\ (s,J)}} \left(\bigoplus_{u \in F^{(s,J)}} M^{gen}_{us} \right) \quad \text{and}$$

$$\dim(M^{gen}_{us}) = \dim(M^{gen}_{vs}) \quad \text{if } u, v \in F^{(s,J)}.$$

Calibrated H-modules

An H-module M is *calibrated* if $\mathbb{C}[Y]$ acts semisimply i.e. ~~dim~~ $(M_\gamma^{gen}) = M_\gamma$.

For $i,j \in \{1,\dots,n\}$ let $R_{ij}^\vee = R^\vee \cap \{a\alpha_i^\vee + b\alpha_j^\vee \mid a,b \in \mathbb{Z}\}$.

A *skew local region* is a local region (s,J) such that

(a) If $w \in \mathcal{F}^{(s,J)}$ then $(ws)(y^{\alpha_i^\vee}) \neq 1$.

(b) If $w \in \mathcal{F}^{(s,J)}$ and $i,j \in \{1,\dots,n\}$ and $Z(ws) \cap R_{ij}^\vee \neq \emptyset$ then $\mathrm{Card}(P(ws) \cap R_{ij}^\vee) > 2$

Theorem

$$\left\{\begin{matrix}\text{calibrated simple}\\ \text{H-modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{skew local regions}\\ (s,J)\end{matrix}\right\}$$

$$H^{(s,J)} \longleftarrow (s,J)$$

and $\dim(H^{(s,J)}) = \mathrm{Card}(\mathcal{F}^{(s,J)})$

and $\dim(H^{(s,J)}_{ws}) = \begin{cases} 1, & \text{if } i \in \mathcal{F}^{(s,J)} \\ 0, & \text{otherwise} \end{cases}$

Note: If s ~~is~~ regular then M is calibrated.

There are ~~many~~ non regular calibrated simple Modules.

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The case GL_n

$P^\vee$ has $\mathbb{Z}$-basis $\{\varepsilon_1^\vee, \dots, \varepsilon_n^\vee\}$

$W_0 = S_n$ permutes $\varepsilon_1^\vee, \dots, \varepsilon_n^\vee$

$\alpha_i^\vee = \varepsilon_{i+1}^\vee - \varepsilon_i^\vee$, for $i \in \{1, \dots, n-1\}$.

$(R^\vee)^+ = \{\varepsilon_j^\vee - \varepsilon_i^\vee \mid 1 \le i < j \le n\}$.

The torus $T^\vee = \mathrm{Hom}(P^\vee, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^n$

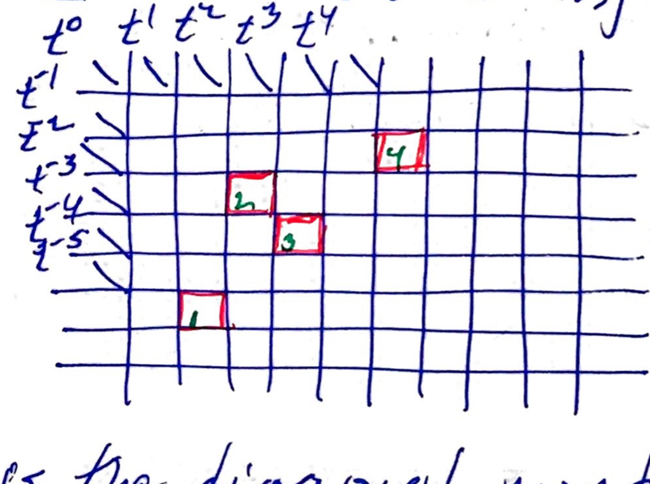
$$\gamma : \mathbb{C}[Y] \to \mathbb{C}, \qquad y^{\varepsilon_i^\vee} \mapsto \gamma_i \qquad \text{with } (\gamma_1, \dots, \gamma_n) \in (\mathbb{C}^\times)^n$$

Write
$$\gamma = (\gamma_1, \dots, \gamma_n) = (t^{c_1}, t^{c_2}, \dots, t^{c_n})$$

Assume $c_1, \dots, c_n \in \mathbb{Z}$ for combinatorial illustration.

A <u>box</u> is an element $b = (i,j) \in \mathbb{Z} \times \mathbb{Z}$.

The <u>content</u> of the box $b = (i,j)$ is $c(b) = j - i$.



$= (t^{-5}, t^0, t^0, t^4) = \gamma \in T^\vee$

$c(b)$ is the diagonal number of b.

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Multisegments

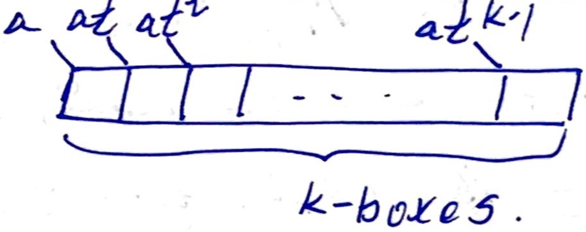
Theorem (Bernstein-Zelevinsky) Type GL_n

$$\left\{\begin{matrix}\text{simple}\\ H\text{-modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{multisegments}\\ \text{with } n \text{ boxes}\end{matrix}\right\}$$

$$H^{(\gamma,J)} \longleftarrow (\gamma,J)$$

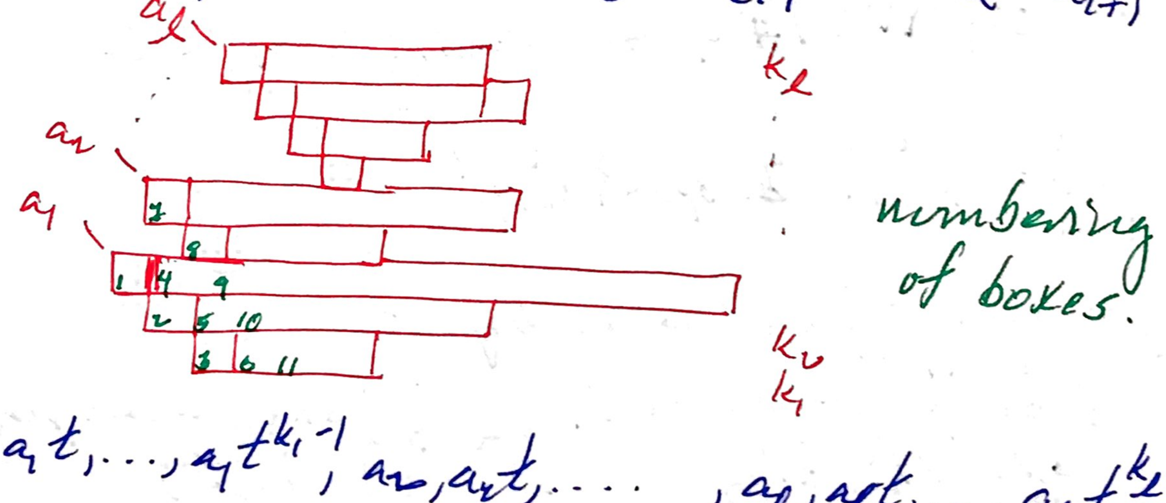
A segment is (a,k) with $a\in\mathbb{C}^\times$, $k\in\mathbb{Z}_{>0}$.

k boxes; first box is content a.



A multisegment with n boxes is a multiset $\{(a_1,k_1),\ldots,(a_\ell,k_\ell)\}$ with $k_1+k_2+\cdots+k_\ell=n$.

Assume $a_1,\ldots,a_\ell\in t^{\mathbb{Z}}$ and order the segments $a_1\le a_2\le\cdots\le a_\ell$ and $k_i\le k_{i+1}$ if $a_i=a_{i+1}$



$\gamma=(a_1,a_1t,\ldots,a_1t^{k_1-1},a_2,a_2t,\ldots,a_\ell,a_\ell t,\ldots,a_\ell t^{k_\ell-1})$.

$J=\{\varepsilon_j^\vee-\varepsilon_i^\vee \mid j>i$, box$_j$ and box$_i$ are in adjacent diagonals, box$_j$ is strictly north and weakly west of box$_i\}$