Representations of affine Hecke algebras III

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The groups W_K

$$W = \{ w t_{\lambda^\vee} \mid w \in W_0, \lambda^\vee \in P^\vee \} \qquad \left(\begin{array}{c}\text{affine}\\ \text{Weyl}\\ \text{group}\end{array}\right)$$

W_0 is a finite group acting on

$$P^\vee = \mathbb{Z}\text{-span}\{\omega_1^\vee, \ldots, \omega_n^\vee\}$$

generated by $s_1, \ldots, s_n$ where

$$s_i \lambda^\vee = \lambda^\vee - \lambda_i \alpha_i^\vee \quad \text{if } \lambda = \lambda_1 \omega_1^\vee + \cdots + \lambda_n \omega_n^\vee.$$

For $K \subseteq \{1, \ldots, n\}$ define

$$W_K = \langle s_k \mid k \in K \rangle \subseteq W_0.$$

The subalgebras H_K

H has basis $\{ T_w y^{\lambda^\vee} \mid \lambda^\vee \in P^\vee, w \in W_0 \}$

and subalgebras

$$\mathbb{C}[Y] = \text{span}\{ y^{\lambda^\vee} \mid \lambda^\vee \in P^\vee \} = \mathbb{C}[y^{\pm\omega_1^\vee}, \ldots, y^{\pm\omega_n^\vee}]$$

$$H_{fin} = \text{span}\{ T_w \mid w \in W_0 \} \text{ with } (T_{s_i} - t^{\frac12})(T_{s_i} + t^{-\frac12}) = 0$$

$$H_K = \text{span}\{ T_w y^{\lambda^\vee} \mid w \in W_K, \lambda^\vee \in P^\vee \}.$$

Let $T^\vee = \text{Hom}(\mathbb{C}[Y], \mathbb{C}) = (\mathbb{C}^\times)^n$

$$\begin{aligned} \gamma : \mathbb{C}[Y] &\to \mathbb{C} \\ y^{\omega_i^\vee} &\mapsto \gamma_i \end{aligned} \qquad \text{with } (\gamma_1, \ldots, \gamma_n) \in (\mathbb{C}^\times)^n.$$

Standard modules $M^{(\gamma, e_K)}$

Assume $\gamma \in T^\vee$ satisfies

$$\gamma(y^{\alpha_k^\vee}) = t^{-1} \quad \text{for } k \in K.$$

Define a 1-dim'l H_K-module $\mathbb{C}_\gamma = \mathbb{C} v_\gamma$

$$y^{\omega_i} v_\gamma = \gamma_i v_\gamma \quad \text{for } i \in \{1, \dots, n\}$$

$$T_{s_k} v_\gamma = -t^{-\frac{1}{2}} v_\gamma \quad \text{for } k \in K$$

Define

$$M^{(\gamma, e_K)} = \mathrm{Ind}_{H_K}^{H}(\mathbb{C}_\gamma) = H \otimes_{H_K} \mathbb{C}_\gamma.$$

Principal series modules

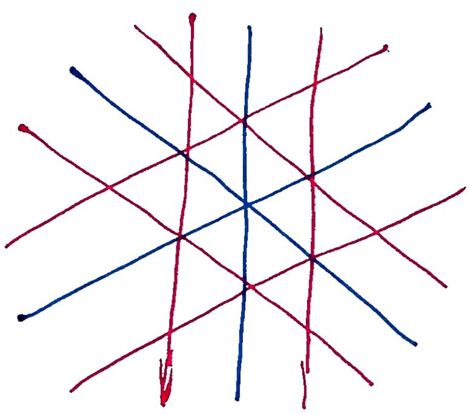
The principal series modules are

$$M^{(\gamma, e_\emptyset)} \quad \text{for } \gamma \in T^\vee.$$

Let $R^\vee = W_0 \cdot \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ and

$$P(\gamma) = \{\beta^\vee \in R^\vee \mid \gamma(y^{\beta^\vee}) \in \{t, t^{-1}\}\}$$

$$= \{\text{red hyperplanes } \gamma \text{ is on}\}.$$



Kato's theorem

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Rep. Thy Sem. 3
A. Ram

Theorem

(a) $M^{(\gamma, e_\phi)}$ is simple $\Leftrightarrow P(\gamma) = \phi$.

(b) Every simple H-module is a quotient of some $M^{(\gamma, e_\phi)}$.

(c) W_0 acts on $T^\vee$ by

$$(w\gamma)(y^{\lambda^\vee}) = \gamma(y^{w^{-1}\lambda^\vee}) \text{ for } w \in W_0, \lambda^\vee \in P^\vee, \gamma \in T^\vee.$$

Let $w \in W_0$. Then $M^{(\gamma, e_\phi)}$ and $M^{(w\gamma, e_\phi)}$ have the same composition factors.

Idea NOT true without more adjectives

$$\left\{\begin{matrix}\text{simple} \\ \text{H-modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Pairs } (\gamma, e_K) \text{ where} \\ K \subseteq \{1, \dots, n\} \text{ and} \\ \gamma(y^{\alpha_k^\vee}) = t^{-1} \text{ and } k \in K\end{matrix}\right\}$$

$$\frac{M^{(\gamma, e_K)}}{\left(\begin{matrix}\text{max. proper} \\ \text{submodule}\end{matrix}\right)} \longleftarrow\!\shortmid (\gamma, e_K)$$

Langlands parameters

For $\gamma \in T^\vee$ let $r(\gamma)_1, \ldots, r(\gamma)_n$ be given by

$$r(\gamma)_i = \mathrm{Re}(l(\gamma)_i) \quad \text{where} \quad \gamma_i = t^{l(\gamma)_i}.$$

Let M be a simple H-module, a quotient of $M(s, e_\phi)$. Then

$$M = \bigoplus_{\gamma \in W_0 s} M_\gamma^{gen}$$

M is <u>tempered</u> if M satisfies

if $M_\gamma^{gen} \neq 0$ then $r(\gamma)_1, \ldots, r(\gamma)_n \leq 0$.

M is <u>square integrable</u> if M satisfies

if $M_\gamma^{gen} \neq 0$ then $r(\gamma)_1, \ldots, r(\gamma)_n < 0$

<u>Theorem</u>

(a) Let M be a simple H-module.
there exists a unique $s \in T^\vee$
$K \subseteq \{1, \ldots, n\}$, and
U_K a simple tempered H_K-module
such that

M is a quotient of $\mathrm{Ind}_{H_K}^{H}(U_K)$.

(b) Let U_K be a simple H_K-module.

Then there is a unique $L \subseteq K$ and

V_L a simple square integrable H_L-module

such that

U_K is a quotient of $\mathrm{Ind}_{H_L}^{H_K}(V_L)$.

(c) If V_L is square integrable then

there does not exist $J \subsetneq L$ and

a simple ~~H_J~~ module W_J such that V_L is

a quotient of $\mathrm{Ind}_{H_J}^{H_J}(W_J)$

Generalized Springer Fibers

$G^\vee$ reductive alg. group with root system $R^\vee$.
$\cup|$
$B^\vee$ Borel subgroup corresp. to $\alpha_1^\vee, \ldots, \alpha_n^\vee$
$\cup|$
$T^\vee = \mathrm{Hom}(P^\vee, \mathbb{C}^\times) =$ maximal torus

Let

$$\mathfrak{g}^\vee = \mathrm{Lie}(G^\vee) \text{ and } \mathfrak{b}^\vee = \mathrm{Lie}(B^\vee)$$

A <u>nilpotent</u> is $e \in \mathfrak{g}^\vee$ such that
there exists $\ell \in \mathbb{Z}_{>0}$ with $(\mathrm{ad}_e)^\ell = 0$

The <u>Springer fiber</u> is

$$\mathcal{B}_e = \{ gB^\vee \mid \mathrm{Ad}_g(e) \in \mathfrak{b}^\vee \} \subseteq G^\vee / B^\vee$$

For $s \in T^\vee$ let

$$\mathcal{B}_e^s = \{ gB^\vee \in \mathcal{B}_e \mid sgB^\vee = gB^\vee \} \quad \left(\begin{array}{l} \text{generalized} \\ \text{Springer fiber} \end{array} \right)$$

<u>Jacobson-Morozov</u> There exists $f, h \in \mathfrak{g}^\vee$
such that

$$[e,f] = h, \ [h,e] = 2e, \ [h,f] = -2f.$$

Using $\exp: \mathfrak{g}^\vee \to G^\vee$ let

$$s_e = \exp(\kappa h), \quad \text{where } t = e^\kappa$$

Then $\mathrm{Ad}_{s_e}(e) = te$

Cuspidal nilpotents and square integrable modules

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For $K \subseteq \{1,\ldots,n\}$ let

$G_K^\vee$ be the Levi subgroup corresponding to $\{\alpha_k^\vee \mid k \in K\}$.

$\mathfrak{g}_K^\vee = \mathrm{Lie}(G_K^\vee) \subseteq \mathfrak{g}^\vee$

A <u>cuspidal nilpotent</u> is a nilpotent $e \in \mathfrak{g}^\vee$ such that there <u>does</u> <u>not</u> <u>exist</u>

$K \subseteq \{1,\ldots,n\}$ with $e \in \mathfrak{g}_K^\vee$.

<u>Theorem</u> Let $e \in \mathfrak{g}^\vee$ be nilpotent.
Let $s \in T^\vee$ such that $\mathrm{Ad}_s(e) = te$.

(a) $K(\mathcal{B}_e^s)$ is an H-module.

(b) Let $K \subseteq \{1,\ldots,n\}$ and $e_K \in \mathfrak{g}_K^\vee$ nilpotent.
Let $s \in T^\vee$ such that $\mathrm{Ad}_s(e_K) = te_K$
Then
$$K(\mathcal{B}_{e_K}^s) = \mathrm{Ind}_{H_K}^H (K(\mathcal{B}_{e_K}^s))$$

(c) $K(\mathcal{B}_e^{se})$ is a square integrable H-module if and only if
e is a cuspidal nilpotent

(d) Every simple H-module is a quotient of some $K(\mathcal{B}_e^s)$