15.06.2021

Representations of affine Hecke algebras

1

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A a (fin. dim'l semisimple) algebra

M an A-module

$Z = \mathrm{End}_A(M)$ the centralizer algebra

As an (A, Z)-bimodule

$$M \simeq \bigoplus_{\lambda \in \hat{Z}} A^\lambda \otimes Z^\lambda$$

and

$$\left\{\begin{matrix}\text{simple } A\text{-modules} \\ \text{which appear in } M\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{simple} \\ Z\text{-modules}\end{matrix}\right\}$$

$$M \otimes_Z Z^\lambda = A^\lambda \longleftrightarrow Z^\lambda = A^\lambda \otimes_A M$$

$$\mathrm{Hom}_Z(M, Z^\lambda) = A^\lambda \longleftrightarrow Z^\lambda = \mathrm{Hom}_A(A^\lambda, M)$$

<u>Example</u> G a (finite) group

$$A = \mathbb{C}G = \mathrm{span}\{g \in G\}$$

$$M = \mathbb{C}G = \mathrm{span}\{m \in G\}.$$

Then

$$\mathrm{End}_A(M) = Z \xrightarrow{\sim} \mathbb{C}G$$

$$\begin{matrix}\varphi_g : M \to M \\ m \mapsto mg\end{matrix} \longleftarrow g$$

$$\varphi \longmapsto \varphi(1).$$

15.06.2021 ②
Rep Thy Sem.
A. Ram

<u>The Hecke algebra of (G,B)</u>

G a (finite) group $\supseteq$ B a subgroup.

$A = \mathbb{C}G$

$M = \mathrm{Ind}_B^G(\mathbb{1}_B) = \mathbb{C}G \otimes_{\mathbb{C}B} \mathbb{1}_B = (\mathbb{C}G)\mathbb{1}_B = A\mathbb{1}_B$

with $b\mathbb{1}_B = \mathbb{1}_B$ for $b \in B$.

Then

$$M = \mathbb{C}\text{-span}\left\{ y\mathbb{1}_B \mid y \in \left\{\begin{matrix}\text{reps of cosets}\\ \text{in } G/B\end{matrix}\right\}\right\}$$

$$= \mathbb{C}\text{-span}\left\{ m_y \mid y \in \left\{\begin{matrix}\text{reps of cosets}\\ \text{in } G/B\end{matrix}\right\}\right\}$$

where

$$m_y = \sum_{x \in yB} x = y\left(\sum_{b\in B} b\right) = y\mathbb{1}_B.$$

$$G = \bigsqcup_{w \in W_0} BwB, \quad \text{where } W_0 \text{ is } \left\{\begin{matrix}\text{reps of}\\ \text{double cosets}\\ \text{in } B\backslash G/B\end{matrix}\right\}$$

Let $$T_w = \sum_{x \in BwB} x = \mathbb{1}_B w \mathbb{1}_B.$$

The <u>Hecke algebra</u> is

$$\mathbb{C}[B\backslash G/B] = \mathbb{C}\text{-span}\{T_w \mid w \in W_0\}.$$

H is a (nonunital) subalgebra of $\mathbb{C}G$

The identity in H is $T_1 = \mathbb{1}_B \mid \mathbb{1}_B = \mathbb{1}_B$.

$$\mathrm{End}_A(M) = Z \xrightarrow{\sim} H = \mathbb{C}[B\backslash G/B]$$

$$\begin{aligned} \varphi_w : M &\longrightarrow M & &\longleftarrow\!\shortmid\ T_w \\ m_y &\longmapsto m_y T_w & & \end{aligned}$$

$$\varphi \longmapsto \varphi(m_1).$$

and

$$\left\{\begin{matrix}\text{simple } G\text{-modules} \\ \text{generated by } V^B\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{simple} \\ H\text{-modules}\end{matrix}\right\}$$

$$V \longmapsto V^B = \{v \in V \mid bv = v \text{ for } b \in B\}$$

$$M \otimes_H Z^\lambda \longleftarrow\!\shortmid\ Z^\lambda$$

(see Borel, Inv. Math $\underline{35}$ (1976) 233–259)

15.06.2021
Rep. Thy Talk

Convolution

Use

$$\mathbb{C}G \xrightarrow{\sim} \mathcal{C}_G = \{f : G \to \mathbb{C}\}$$

$$\sum_{g\in G} f(g) g \longleftarrow\!\shortmid f$$

$$g \longmapsto \delta_g : \mathbb{C} \to \mathbb{C} \text{ with } \delta(x) = \begin{cases} 1, & x = g, \\ 0, & x \neq g. \end{cases}$$

to replace $\mathbb{C}G$ by $\mathcal{C}_G$. The product in $\mathcal{C}_G$ is

$$(f_1 * f_2)(x) = \sum_{g\in G} f_1(xg^{-1}) f_2(g)$$

$$= \int_G f_1(xg^{-1}) f_2(g)\, dg$$

Then

$$\mathbb{1}_B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \notin B, \end{cases} \qquad T_w(x) = \begin{cases} 1, & \text{if } x \in BwB, \\ 0, & \text{if } x \notin BwB, \end{cases}$$

and

$$\mathbb{1}_B * \mathbb{1}_B = \mathbb{1}_B \quad \text{since} \quad \int_G \mathbb{1}_B(g)\, dg = 1.$$

B-cosets in $G = GL(\mathbb{F}_p)$

$$G = GL_3(\mathbb{F}_p) \supseteq B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

Let

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and for $c \in \mathbb{F}_p$ let

$$y_1(c) = \begin{pmatrix} c & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad y_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Then

$$G = B1B \sqcup Bs_1B \sqcup Bs_2B \sqcup Bs_1s_2B \sqcup Bs_2s_1B \sqcup Bs_1s_2s_1B.$$

with

	# of B-cosets
$B1B = B$	1
$Bs_1B = \bigsqcup_{c \in \mathbb{F}_p} y_1(c)B$	p
$Bs_2B = \bigsqcup_{c \in \mathbb{F}_p} y_2(c)B$	p
$Bs_1s_2B = \bigsqcup_{c_1, c_2 \in \mathbb{F}_p} y_1(c_1)y_2(c_2)B$	p^2
$Bs_2s_1B = \bigsqcup_{c_1, c_2 \in \mathbb{F}_p} y_2(c_1)y_1(c_2)B$	p^2
$Bs_1s_2s_1B = \bigsqcup_{c_1, c_2, c_3 \in \mathbb{F}_p} y_1(c_1)y_2(c_2)y_1(c_3)B$	p^3

and

$$\mathrm{Card}(G/B) = 1 + 2p + 2p^2 + p^3 = (1+p)(1+p+p^2) = \frac{(1-p)}{(1-p)}\frac{(1-p^2)}{(1-p)}\frac{(1-p^3)}{(1-p)}$$

15.06.2021 ⑥
Rep Thy Sem.
A. Ram

<u>p-adic groups $G(\mathbb{Q}_p)$</u>

$$\mathbb{Q}_p = \left\{ a_\ell p^\ell + a_{\ell+1} p^{\ell+1} + \cdots \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z} \\ a_i \in \mathbb{F}_p \\ a_\ell \neq 0 \end{array} \right\} \cup \{0\}$$

$$\cup\!|$$

$$\mathbb{Z}_p = \left\{ a_\ell p^\ell + a_{\ell+1} p^{\ell+1} + \cdots \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0} \\ a_i \in \mathbb{F}_p \\ a_\ell \neq 0 \end{array} \right\} \cup \{0\}.$$

$$\cup\!|$$

$$\mathbb{Z} = \left\{ a_\ell p^\ell + a_{\ell+1} p^{\ell+1} + \cdots \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0},\ a_i \in \mathbb{F}_p,\ a_\ell \neq 0 \\ \text{all but a finite} \\ \text{number of } a_i \text{ are } 0 \end{array} \right\} \cup \{0\}$$

i.e. $\mathbb{Q}_p = \mathbb{F}_p((p))$, $\mathbb{Z}_p = \mathbb{F}_p[[p]]$ and $\mathbb{Z} = \mathbb{F}_p[p]$ (with a slightly different multiplication).

$$\begin{array}{ccc} G = GL_n(\mathbb{Q}_p) & & \\ \cup\!| & & \\ K = GL_n(\mathbb{Z}_p) & \xrightarrow[p=0]{\Phi} & GL_n(\mathbb{F}_p) \\ (k_{ij}) & \longmapsto & (k_{ij} \bmod p) \\ \cup\!| & & \cup\!| \\ I = \Phi^{-1}(B) & \longrightarrow & B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \end{array}$$

i.e.

$$I = \left\{ \begin{pmatrix} a_{11} & & a_{ij} \\ & \ddots & \\ a_{ji} & & a_{nn} \end{pmatrix} \;\middle|\; \begin{array}{l} a_{ij} \in \mathbb{Z}_p \\ a_{ii} \in \mathbb{Z}_p^\times \\ a_{ji} \in p\mathbb{Z}_p \end{array} \right\}$$

The <u>affine Hecke algebra</u> is the Hecke algebra for (G, I)

$$H = \mathbb{C}[I \backslash G / I] = \mathbb{C}\text{-span}\{T_w \mid w \in W\}$$

with $W = \left\{ \begin{matrix} \text{representatives of double} \\ \text{cosets in } I\backslash G/I \end{matrix} \right\}$

i.e. $G = \bigsqcup_{w \in W} IwI$.

<u>Example</u> $G = GL_n(\mathbb{Q}_p)$. Let

$W_0 = S_n$ and $P^\vee = \mathbb{Z}^n = \{\mu = (\mu_1, \ldots, \mu_n) \mid \mu_i \in \mathbb{Z}\}$

Let $t_\mu = \begin{pmatrix} p^{\mu_1} & & 0 \\ & p^{\mu_2} & \\ & \ddots & \\ 0 & & p^{\mu_n} \end{pmatrix}$ and

$$W = \{t_\mu w \mid w \in S_n \text{ and } \mu \in \mathbb{Z}^n\}$$

<u>Idea</u>: If $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word then

$$IwI = \bigsqcup_{c_1, \ldots, c_\ell \in \mathbb{F}_p} y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) I$$

I-cosets in $G(\mathbb{Q}_p) = G$

G is generated by SL_2s corresponding to the nodes of the affine Dynkin diagram

$$\begin{aligned}
\varphi_i : SL_2(\mathbb{F}_p) &\longrightarrow G \\
\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} &\longmapsto x_{\alpha_i}(c) \\
\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} &\longmapsto x_{-\alpha_i}(c) \\
\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} &\longmapsto h_{\alpha_i^\vee}(d) \\
\begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix} &\longmapsto y_i(c)
\end{aligned}$$

Let $\tilde{s}_i = y_i(0)$.

Then

$$y_i(c_1)y_i(c_2) = \begin{cases} y_i(c_1 - c_2^{-1})\, h_{\alpha_i^\vee}(c_2)\, x_{\alpha_i}(-c_2^{-1}), & \text{if } c_2 \neq 0 \\ h_{\alpha_i^\vee}(-1)\, x_{\alpha_i}(c_1), & \text{if } c_2 = 0 \end{cases}$$

Coxeter like relns

If $\overset{i}{\circ}\quad\overset{j}{\circ}$ then $y_i(c_1)y_j(c_2) = y_j(c_2)y_i(c_1)$

If $\overset{i}{\circ}\!-\!\overset{j}{\circ}$ then $y_i(c_1)y_j(c_2)y_i(c_3)$

$$= y_j(c_3)\, y_i(c_1c_3 - c_2)\, y_j(c_1)$$

If $\overset{i}{\circ}\!=\!\overset{j}{\circ}$ then $y_i(c_1)y_j(c_2)y_i(c_3)y_j(c_4)$

$$= y_j(-c_4)\, y_i(-c_1c_4^2 - 2c_2c_4 + c_3)\, y_j(c_1c_4 + c_2)\, y_i(c_1)$$

Then

$$I = \left\langle h_{\lambda^\vee}(d),\ x_{\alpha_i}(c) \;\middle|\; \begin{matrix}\lambda^\vee \in P^\vee,\ d \in \mathbb{F}_p^\times \\ i \in \{0,1,\ldots,n\},\ c \in \mathbb{F}_p\end{matrix} \right\rangle$$

$$G = \langle I,\ y_i(c) \mid i \in \{0,1,\ldots,n\},\ c \in \mathbb{F}_p \rangle.$$

Since

$$x_\beta(d)\, y_i(c) = y_i(c)\, x_{s_i\beta}(d) \quad \text{if } \beta \in R^+ \text{ and } \beta \neq \alpha_i$$

$$x_{\alpha_i}(c_1)\, y_i(c_2) = y_i(c_1 + c_2)$$

$$h_{\lambda^\vee}(d)\, y_i(c) = y_i\big(d^{\langle \lambda^\vee, \alpha_i\rangle} c\big)\, h_{s_i\lambda^\vee}(d)$$

which give: If $b_1 \in I$ and $c_2 \in \mathbb{F}_p$ then

$$b_1\, y_i(c_2) = y_i(\tilde{c}_2)\, \tilde{b}_1$$

for unique $\tilde{c}_2 \in \mathbb{F}_p$ and $\tilde{b}_1 \in I$.

<u>The Hecke relation</u>

Suppose $\tilde{T}_{s_i}^2 = (p^{\frac12} - p^{-\frac12})\tilde{T}_{s_i} + 1$.

Put $T_{s_i} = p^{\frac12}\tilde{T}_{s_i}$.

Then

$$\begin{aligned} T_{s_i}^2 = p\,\tilde{T}_{s_i}^2 &= p\big((p^{\frac12} - p^{-\frac12})\tilde{T}_{s_i} + 1\big) \\ &= p\big((p^{\frac12} - p^{-\frac12})p^{-\frac12} T_{s_i} + 1\big) \\ &= (p-1)T_{s_i} + p. \end{aligned}$$

15.06.2021
Rep Thy Seminar
A. Ram

(10)

$$T_{s_i} T_{s_i} = \Big(\sum_{x \in I s_i I} x\Big)\Big(\sum_{y \in I s_i I} y\Big)$$

$$= \Big(\sum_{\substack{c_1 \in \mathbb{F}_p \\ b_1 \in I}} y_i(c_1) b_1\Big)\Big(\sum_{\substack{c_2 \in \mathbb{F}_p \\ b_2 \in I}} y_i(c_2) b_2\Big)$$

$$= \sum_{\substack{c_1, c_2 \in \mathbb{F}_p \\ b_1, b_2 \in I}} y_i(c_1) b_1 y_i(c_2) b_2 = \sum_{\substack{c_1, \tilde{c}_2 \in \mathbb{F}_p \\ \tilde{b}_1, b_2 \in I}} y_i(c_1) y_i(\tilde{c}_2) \tilde{b}_1 b_2$$

$$= \sum_{\substack{c_1 \in \mathbb{F}_p \\ \tilde{c}_2 \in \mathbb{F}_p^\times \\ \tilde{b}_1, b_2 \in I}} y_i(c_1 - \tilde{c}_2^{-1}) h_{\alpha_i^\vee}(\tilde{c}_2) x_{\alpha_i}(-\tilde{c}_2^{-1}) \tilde{b}_1 b_2$$

$$+ \sum_{\substack{c_1 \in \mathbb{F}_p \\ \tilde{b}_1, b_2 \in I}} h_{\alpha_i^\vee}(-1) x_{\alpha_i}(c_1) \tilde{b}_1 b_2$$

$$= (p-1) T_{s_i} \cdot T_1 + p\, T_1 \cdot T_1$$

$$= (p-1) T_{s_i} + p\, T_1$$