16.06.2021 ①
Solvable Lattice Models Talk
A. Ram

Level 0 modules for quantum affine algebras

Which modules have crystals?

$\mathfrak{g}$ = affine Lie algebra U = quantum affine algebra

$\cup\!\!|$

$\mathring{\mathfrak{g}}$ fin. dim'l Lie algebra $\mathring{U}$ = quantum group

Integrable modules have crystals!

How do you index integrable modules?

$\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ fundamental weights for $\mathfrak{g}$

$\omega_1, \ldots, \omega_n$ fundamental weights for $\mathring{\mathfrak{g}}$

$$\Lambda_i = \omega_i + \Lambda_0 \quad \text{for } i \in \{1, \ldots, n\}.$$

Integrable modules $L(\lambda)$ indexed by $\mathfrak{h}^{int}$

$$\mathfrak{h}^{int} = \mathfrak{h}^+_{\mathbb{Z}} \cup \mathfrak{h}^0_{\mathbb{Z}} \cup \mathfrak{h}^-_{\mathbb{Z}}$$

$$\mathfrak{h}^+_{\mathbb{Z}} = \mathbb{Z}_{\geq 0}\text{-span}\{\Lambda_0, \Lambda_1, \ldots, \Lambda_n\}$$

$$\mathfrak{h}^0_{\mathbb{Z}} = \mathbb{Z}_{\geq 0}\text{-span}\{\omega_1, \ldots, \omega_n\}$$

$$\mathfrak{h}^-_{\mathbb{Z}} = \mathbb{Z}_{\leq 0}\text{-span}\{\Lambda_0, \Lambda_1, \ldots, \Lambda_n\}.$$

Extremal weight modules $L(\lambda)$

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$\lambda = \ell\Lambda_0 + m_1\omega_1 + \cdots + m_n\omega_n \in \mathfrak{h}^{int}$.

ℓ is the level of λ (and $L(\lambda)$).

Presentation of $L(\lambda)$: W is the affine Weyl group.

Generators: $\{m_{w\lambda} \mid w \in W\}$

Relations:

$$c^{\frac12} m_{w\lambda} = m_{w\lambda} t^{\frac12 \ell}, \quad K_i m_{w\lambda} = t^{\langle w\lambda, \alpha_i^\vee\rangle} m_{w\lambda}$$

If $\langle w\lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$ then

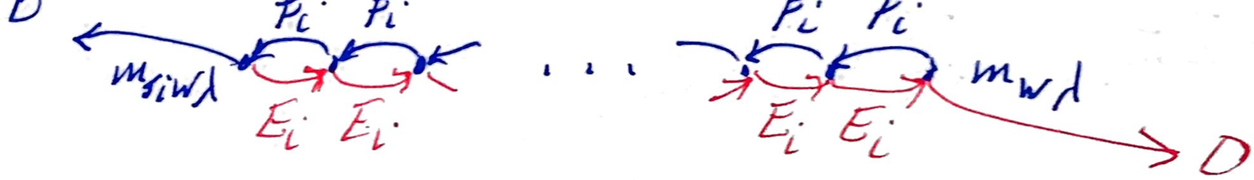
$$E_i m_{w\lambda} = 0 \quad \text{and} \quad F_i^{\langle w\lambda, \alpha_i^\vee\rangle} m_{w\lambda} = m_{s_i w\lambda}$$

If $\langle w\lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\leq 0}$ then

$$F_i m_{w\lambda} = 0 \quad \text{and} \quad E_i^{-\langle w\lambda, \alpha_i^\vee\rangle} m_{w\lambda} = m_{s_i w\lambda}$$

Picture:

If $\langle w\lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$ then



Loop presentation for $L(\lambda)$ when $\lambda \in \mathfrak{h}_{\mathbb{Z}}^{0}$

$$\lambda = m_1\omega_1 + \cdots + m_n\omega_n$$

Generator: m_λ

Relations:

$$c^{\frac{1}{2}} m_\lambda = m_\lambda, \qquad K_i m_\lambda = t^{m_i} m_\lambda$$

$x^+_{i,r} m_\lambda = 0$ for $i \in \{1, \ldots, n\}$ and $r \in \mathbb{Z}$

$e^{(i)}_s m_\lambda = 0$ for $i \in \{1, \ldots, n\}$ and $s \in \mathbb{Z}_{>m_i}$

$e^{(i)}_{-s} m_\lambda = 0$ for $i \in \{1, \ldots, n\}$ and $s \in \mathbb{Z}_{>m_i}$.

Let

$$e^{(i)}_+(u) = 1 + \sum_{s \in \mathbb{Z}_{>0}} e^{(i)}_s u^s$$

$$e^{(i)}_-(u) = 1 + \sum_{s \in \mathbb{Z}_{>0}} e^{(i)}_{-s} u^s.$$

Finite dimensional modules

A Drinfeld polynomial is an element of $\mathbb{C}[u]\text{-span}\{\omega_1, \ldots, \omega_n\}$,

$$a(u) = a^{(1)}(u)\omega_1 + \cdots + a^{(n)}(u)\omega_n.$$

Let

$$\lambda = m_1\omega_1 + \cdots + m_n\omega_n \quad \text{with } m_i = \deg(a^{(i)}(u))$$

The local Weyl module $M^{fin}(a(u))$ is $L(\lambda)$ with

$$e_+^{(i)}(u)m_\lambda = a^{(i)}(u)m_\lambda.$$

Finite dimensional simple modules $L^{fin}(a(u))$ (always level 0)

$$\left\{\begin{matrix}\text{simple fin dim'l} \\ U\text{-modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Drinfeld polynomials} \\ a(u) \in \mathbb{C}[u]\text{-span}\{\omega_1, \ldots, \omega_n\}\end{matrix}\right\}$$

$$L^{fin}(a(u)) \longleftarrow\!\shortmid \quad a(u)$$

where $L^{fin}(a(u)) = \dfrac{M^{fin}(a(u))}{\left(\begin{matrix}\text{max. proper} \\ \text{submodule}\end{matrix}\right)}$

Crystals and Characters

$\lambda = m_1\omega_1 + \cdots + m_n\omega_n$

$B(\lambda)$ the crystal of $L(\lambda)$

$B^{fin}(\lambda)$ the crystal of $M^{fin}(a(\lambda))$

$$S^\lambda = \left\{ \vec{K} = (K^{(1)}, \dots, K^{(n)}) \;\middle|\; \begin{array}{l} K^{(i)} \text{ is a partition} \\ \text{with } \ell(K^{(i)}) < m_i \end{array} \right\}$$

Theorem (Periodicity)

$$B(\lambda) = B^{fin}(\lambda) \times \mathbb{Z}^k \times S^\lambda$$

Where

$k = \#\{m_i \text{ which are nonzero}\}$

$$\mathrm{char}(\mathbb{Z}^k \times S^\lambda) = \prod_{i=1}^{n} \left(\left(\sum_{r \in \mathbb{Z}} q^{r m_i} \right) \left(\prod_{k=1}^{m_i - 1} \frac{1}{1-q^k} \right) \right)$$

$$\mathrm{char}(B^{fin}(\lambda)) = E_{w_0\lambda}(q, 0)$$

where $E_\mu(q, 0)$ denotes the nonsymmetric Macdonald polynomial specialized at $t = 0$.

What is it good for?

$\mathcal{C}$ the category of appadj level 0 modules

$\mathcal{O}$ the category of AppAdj highest wt. modules.

Let V be a finite dimensional module generated by m_ν.

$$V_{aff} = \mathbb{C}[z, z^{-1}] \otimes V$$

(see Hong-Kang page 233)

Vertex operators

Let $\mu, \lambda \in \mathfrak{h}^{+}_{\mathbb{Z}}$.

(Date-Jimbo-Miwa 1993 (3.2)
Frenkel-Reshetikhin 1991 (4.46))

$$\Phi^{V}_{\mu\lambda}(z) : L(\lambda) \longrightarrow L(\mu) \hat{\otimes} V_{aff}$$

i.e. we are studying the action of $\mathcal{C}$ on $\mathcal{O}$

$$\mathcal{O} \times \mathcal{C} \longrightarrow \mathcal{O}.$$

Fock space

(Kashiwara-Miwa-Peterson-Jung 1996)

Wedge products in $\mathcal{C}$ $\qquad \Lambda^n V_{aff} = \dfrac{V_{aff}^{\otimes n}}{N_n}$

Fock spaces in $\mathcal{C}$ $\qquad \mathcal{F}_m = \Lambda^{\infty/2} V_{aff}$

where

$$N_n = \sum_{k=0}^{n-2} V_{aff}^{\otimes k} \otimes N \otimes V_{aff}^{\otimes(n-k-2)}$$

with

$$N = U[z\otimes z, z^{-1}\otimes z^{-1}, z\otimes 1 + 1\otimes z](m_v \otimes m_v)$$

$$= \ker((R-1)\Psi)$$

where Ψ is the factor that removes denominators from R.