(q,t)-Weyl character formula

21.04.2021
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Operators

The symmetric group S_n acts on $\mathbb{C}[x_1,\ldots,x_n]$ by permuting $x_1,\ldots,x_n$.

$$(s_i f)(x_1,\ldots x_n) = f(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_n)$$

for $i\in\{1,\ldots,n-1\}$.

Define $\partial_i : \mathbb{C}[x_1,\ldots,x_n]\to\mathbb{C}[x_1,\ldots,x_n]$ by

$$\partial_i f = \frac{f-s_i f}{x_i-x_{i+1}} \qquad \text{for } i\in\{1,\ldots,n-1\}.$$

Define $T_{s_i}^{-1} : \mathbb{C}[x_1,\ldots,x_n]\to\mathbb{C}[x_1,\ldots,x_n]$ by

$$t^{\frac{1}{2}}T_{s_i}^{-1} = \partial_i x_i - t x_i \partial_i, \qquad \text{for } i\in\{1,\ldots,n-1\},$$

and $T_{s_i} : \mathbb{C}[x_1,\ldots,x_n]\to\mathbb{C}[x_1,\ldots,x_n]$ by

$$T_{s_i} = T_{s_i}^{-1} + t^{\frac{1}{2}} - t^{-\frac{1}{2}}, \qquad \text{for } i\in\{1,\ldots,n-1\}.$$

For $w\in S_n$ define

$$T_w = T_{s_{i_1}}\cdots T_{s_{i_\ell}}$$

if $w = s_{i_1}\cdots s_{i_\ell}$ is reduced.

Macdonald polynomials for GL_n

Use the following recursions to compute

$$E_\mu = E_\mu(x_1, \ldots, x_n; q, t) \quad \text{for } \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$$

(0) $E_{(0,0,\ldots,0)} = 1.$

(1) If $\mu_i > \mu_{i+1}$ then

$$E_{s_i\mu} = \left(\partial_i x_i - t x_i \partial_i + \frac{(1-t)\,q^{\mu_i-\mu_{i+1}}t}{1-q^{\mu_i-\mu_{i+1}}t}\right)E_\mu$$

(2) $E_{(\mu_n+1, \mu_1, \ldots, \mu_{n-1})} = q^{\mu_n} x_1 E_\mu(x_2, \ldots, x_n, q^{-1}x_1).$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$P_\lambda(q,t) = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w\left(E_\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}\right)$$

where $\frac{1}{W_\lambda(t)}$ makes the coefficient of

$x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in $P_\lambda(q,t)$ equal to 1.

The (q,t) Weyl character formula

Let $\delta = (n-1, n-2, \ldots, 1, 0)$ so that

$$\lambda+\delta = (\lambda_1+n-1, \lambda_2+n-2, \ldots, \lambda_{n-1}+1, \lambda_n).$$

Let

$$A_\delta = \prod_{i<j}(x_i - t x_j), \qquad a_\delta = \prod_{i<j}(x_i - x_j)$$

Define

$$A_{\lambda+\delta}(q,t) = \frac{A_\delta}{a_\delta} \sum_{w\in S_n} (-1)^{\ell(w)}\, w E_{\lambda+\delta}$$

Theorem (q,t) Weyl character formula

(a) $A_{0+\delta}(q,t) = \prod_{i<j}(x_i - t x_j) = A_\delta.$

(b) $$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}.$$

Find a proof.

(0,0) Boson-Fermion Correspondence

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$$p_0 = \sum_{w \in S_n} w \qquad e_0 = \sum_{w \in S_n} \det(w)\, w$$

The <u>monomial symmetric functions</u> are

$$m_\lambda = (\text{const})\, p_0 x^\lambda$$

The <u>monomial fermionic functions</u> are

$$a_{\lambda+\delta} = (\text{const})\, e_0 x^{\lambda+\delta}.$$

<u>Master picture</u> $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$

$$\begin{array}{ccc}
\mathbb{C}[X]^{S_n} = p_0 \mathbb{C}[X] & \xrightarrow{\sim} & e_0 \mathbb{C}[X] = a_\delta \mathbb{C}[X]^{S_n} \\
f & \longmapsto & a_\delta f \\
m_\lambda & & \\
\text{Schur function } s_\lambda & \longleftarrow\!\shortmid & a_{\lambda+\delta}
\end{array}$$

The <u>Kostka numbers</u> are $K_{\lambda\mu}$ given by

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

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(q,t) Boson-Fermion Correspondence

The <u>bosonic and fermionic symmetrizers</u>

$$\mathbb{1}_0 = \sum_{w \in S_n} t^{\frac{1}{2}\ell(w)} T_w \qquad \varepsilon_0 = \sum_{w\in S_n} (-t^{-\frac{1}{2}})^{\ell(w)} T_w .$$

The <u>symmetric Macdonald polynomials</u> are

$$P_\lambda(q,t) = (\text{const})\, \mathbb{1}_0 E_\lambda$$

The <u>fermionic Macdonald polynomials</u> are

$$A_{\lambda+\delta}(q,t) = (\text{const})\, \varepsilon_0 E_{\lambda+\delta}$$

(q,t) Master Picture

$$\mathbb{C}[X]^{S_n} = \mathbb{1}_0 \mathbb{C}[X] \xrightarrow{\sim} \varepsilon_0 \mathbb{C}[X] = A_\delta \mathbb{C}[X]^{S_n}$$

$$f \longmapsto A_\delta f$$

$$P_\lambda(q,t)$$

$$P_\lambda(q,qt) \longleftrightarrow A_{\lambda+\delta}(q,t)$$

Define the <u>(q,t) Kostka numbers</u> $K_{\lambda\mu}(q,t)$ by

$$P_\lambda(q,qt) = \sum_\mu K_{\lambda\mu}(q,t) P_\mu(q,t).$$

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(0,t) Boson-Fermion correspondence

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<u>Proposition</u> If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ then

$$E_\lambda(0,t) = x^\lambda.$$

The <u>Hall-Littlewood polynomials</u> are

$$P_\lambda(0,t) = (\text{const})\, \mathbb{1}_0 x^\lambda.$$

The <u>spherical Whittaker functions</u> are

$$A_{\lambda+\delta}(0,t) = (\text{const})\, \varepsilon_0 x^{\lambda+\delta}$$

<u>(0,t) Master picture</u> (Lusztig 1981)

$$\begin{array}{ccc}
\mathbb{C}[X]^{S_n} = \mathbb{1}_0 \mathbb{C}[X] & \xrightarrow{\sim} & \varepsilon_0 \mathbb{C}[X] = A_\delta \mathbb{C}[X]^{S_n} \\
f & \longmapsto & A_\delta f \\
P_\lambda(0,t) & & \\
s_\lambda = P_\lambda(0,0) & \longmapsto & A_{\lambda+\delta}(0,t)
\end{array}$$

The <u>Kostka-Foulkes polynomials</u> are $K_{\lambda\mu}(t)$

$$s_\lambda = \sum_\mu K_{\lambda\mu}(t) P_\mu(0,t).$$

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Some properties of $K_{\lambda\mu}(t)$

(1) $K_{\lambda\mu}(t) = t^{\langle \lambda-\mu, \rho^\vee\rangle} P_{x, n_\lambda}(t^{-1})$

$P_{x,n_\lambda}(t)$ is the Kazhdan-Lusztig polynomial for the affine Weyl group

$x \in W_0 t_\mu W_0$ and n_λ max. length in $W_0 t_\lambda W_0$.

(2) $K_{\lambda\mu}(q^{-1}) = q^{-n(\mu)} \chi_G^\lambda(u_\mu)$

χ_G^λ is character of $GL_n(\mathbb{F}_q)$ appearing in $\mathrm{Ind}_B^G(\mathbb{1})$

u_μ is unipotent of Jordan form μ.

(3) $K_{\lambda\mu}(t) = \sum_{T \in B(\lambda)} t^{ch(T)}$

$B(\lambda) = \{$column strict tableaux of shape $\lambda\}$

$ch(T)$ is the Lascoux-Schutzenberger charge of T.

Generalize (1), (2), (3) to $K_{\lambda\mu}(q,t)$.