Workshop on Macdonald Polynomials 28.06.2021 (1)

Five (q,t) analogues of Kostka numbers A. Ram

(1) $P_\lambda = \sum_\mu K^{(1)}_{\lambda\mu} m_\mu$ (monomial expansion)

generalizes $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$ $(q=t)$

(2) $e_{\mu'} = \sum_\lambda K^{(2)}_{\lambda'\mu'} P_{\lambda'}$ (vertical strip Pieri rule)

generalizes $e_{\mu'} = \sum_\mu K_{\lambda\mu'} s_\lambda$ $(q=t)$

(3) $g_\mu = \sum_\lambda K^{(3)}_{\lambda\mu} P_\lambda$ (horizontal strip Pieri rule)

generalizes $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$ $(q=t)$

(4) $P_\lambda(q,qt) = \sum_\mu K^{(4)}_{\lambda\mu} P_\mu(q,t)$ (Weyl character formula)

generalizes $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$ $(t=1)$

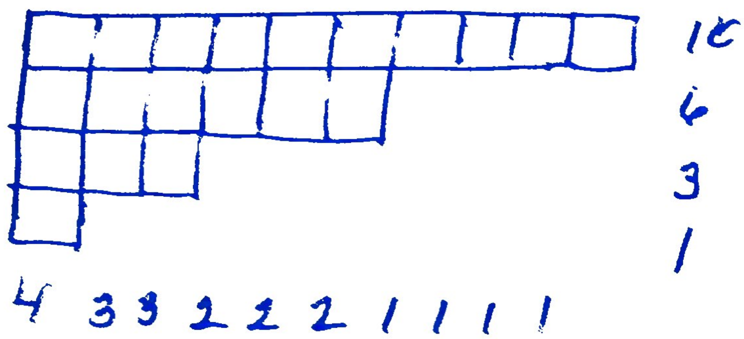
(5) $J_\mu(q,t) = \sum_\lambda K^{(5)}_{\lambda\mu} S_\lambda$ (Macdonald's (q,t)-Kostka)

has $K^{(5)}_{\lambda\mu}(0,1) = K_{\lambda\mu}$

(q, t) hook numbers

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

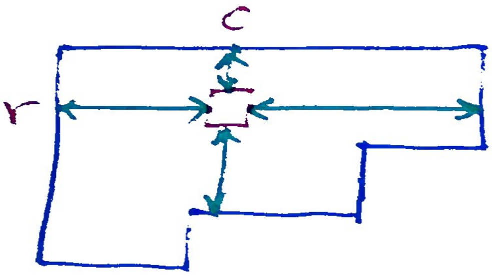
has λ_r boxes in row r, λ'_c boxes in column c.



$\lambda = (10, 6, 3, 1)$

$\lambda' = (4, 3, 3, 2, 2, 2, 1, 1, 1, 1)$.

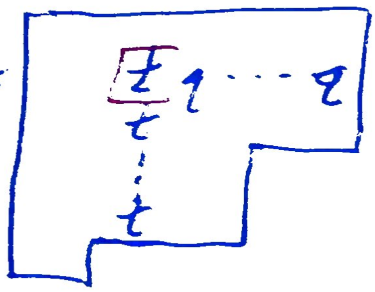
For a box $b = (r, c) \in \lambda$ define



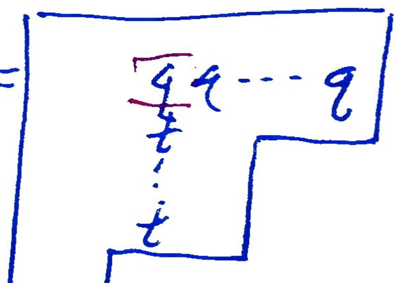
$\mathrm{coleg}_\lambda(b) = c - 1$

$\mathrm{coarm}_\lambda(b) = r - 1 \qquad \mathrm{arm}_\lambda(b) = \lambda_r - r$

$\mathrm{leg}_\lambda(b) = \lambda'_c - c$

$h^\lambda_t(b) =$ 

$$= \begin{cases} 1 - q^{\mathrm{arm}_\lambda(b)} t^{\mathrm{leg}_\lambda(b)+1}, & \text{if } b \in \lambda \\ 1, & \text{if } b \notin \lambda \end{cases}$$

$h^*_\lambda(b) =$ 

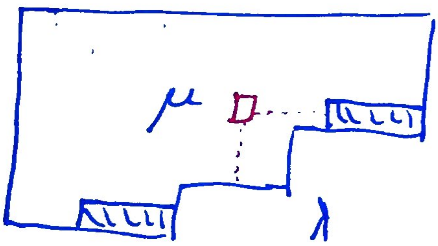
$$= \begin{cases} 1 - q^{\mathrm{arm}_\lambda(b)+1} t^{\mathrm{leg}_\lambda(b)}, & \text{if } b \in \lambda \\ 1, & \text{if } b \notin \lambda \end{cases}$$

$$h^\lambda_t = \prod_{b \in \lambda} h^\lambda_t(b) \qquad h^*_\lambda = \prod_{b \in \lambda} h^*_\lambda(b)$$

$$J_\mu = h^\mu_t P_\mu.$$

Monomial Expansion

A pair of partitions λ/μ is a <u>horizontal strip</u> if $\lambda'_c - \mu'_c \in \{0,1\}$ for $c \in \mathbb{Z}_{>0}$

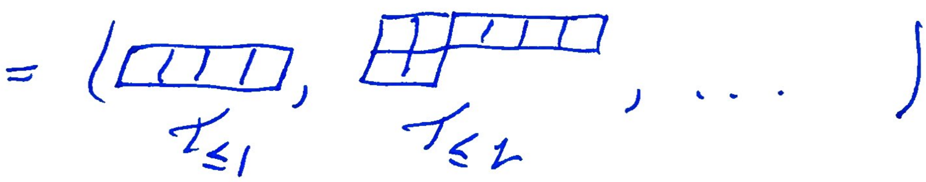


$$\psi_{\lambda/\mu} = \prod_{\substack{b=(r,c)\in\lambda \\ \lambda_r \neq \mu_r \\ \lambda'_c = \mu'_c}} \frac{h^{\mu}_{*}(b)}{h^{*}_{\mu}(b)} \frac{h^{*}_{\lambda}(b)}{h^{\lambda}_{*}(b)}$$

$$B(\lambda)_\nu = \left\{ \begin{matrix} \text{horizontal strip tableaux} \\ \text{of shape } \lambda \text{ and type } \nu \end{matrix} \right\}$$

$$B(\lambda) = \bigcup_\nu B(\lambda)_\nu = \left\{ \begin{matrix} \text{horizontal strip tableaux} \\ \text{of shape } \lambda \end{matrix} \right\}$$

$$T = \begin{matrix} 1&1&1&1&2&2&3&3&3&4 \\ 2&2&3&5&5&5 \\ 3&4&4 \\ 5 \end{matrix} \qquad \begin{matrix} \mathrm{shape}(T) = (10,6,3,1) = \lambda \\ \mathrm{type}(T) = (4,4,5,3,4) = \nu \end{matrix}$$



Let
$$\psi_T = \prod_{i=1}^{n} \psi_{T_{\le i}/T_{<i}}$$

Define
$$P_\lambda(q,t) = \sum_{T \in B(\lambda)} \psi_T x^T$$

where $x^T = x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots x_n^{\#n\text{'s in } T}$

Show that P_λ is a symmetric function.

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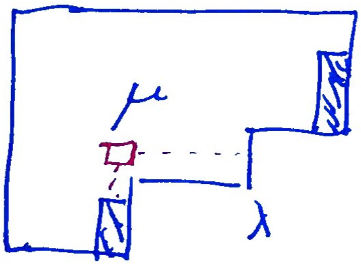
Pieri rules

Define $e_0, e_1, \ldots$ and $g_0, g_1, g_2, \ldots$ by

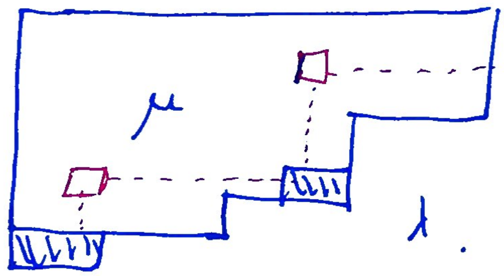
$$\sum_{r=0}^{n} e_r z^r = \prod_{i=1}^{n} (1 + x_i z) \qquad \sum_{r\in\mathbb{Z}_{\geq 0}} g_r z^r = \prod_{i=1}^{n} \frac{(t x_i z; q)_\infty}{(x_i z; q)_\infty}$$

$$e_{\mu'} = e_{\mu'_1} e_{\mu'_2} \cdots \qquad g_\mu = g_{\mu_1} g_{\mu_2} \cdots$$

vertical strip



horizontal strip



$$\psi'_{\lambda/\mu} = \prod_{\substack{b=(r,c)\in\lambda \\ \lambda_r = \mu_r \\ \lambda'_c \neq \mu'_c}} \frac{h_*^\lambda(b)}{h^*_\lambda(b)} \frac{h^*_\mu(b)}{h_*^\mu(b)} \qquad \varphi_{\lambda/\mu} = \prod_{\substack{b=(r,c)\in\lambda \\ \lambda'_c \neq \mu'_c}} \frac{h_*^\lambda(b)}{h^*_\lambda(b)} \frac{h^*_\mu(b)}{h_*^\mu(b)}$$

Define $P_\lambda(q,t)$ by one of the following

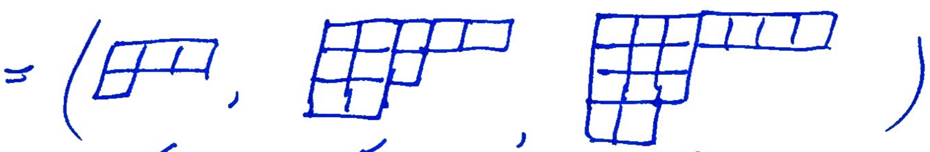
$$e_r P_\mu = \sum_{\substack{\lambda \\ \lambda/\mu \text{ vert. strip}}} \psi'_{\lambda/\mu} P_\lambda \qquad g_r P_\mu = \sum_{\substack{\lambda \\ \lambda/\mu \text{ horiz. strip}}} \varphi_{\lambda/\mu} P_\lambda$$

Big Schurs

$BB(\lambda)_\nu = \left\{ \begin{array}{l} \text{broken border strip tableaux} \\ \text{of shape } \lambda \text{ and type } \nu \end{array} \right\}$

$BB(\lambda) = \bigcup_\nu BB(\lambda)_\nu = \left\{ \begin{array}{l} \text{broken border strip tableaux} \\ \text{of shape } \lambda \end{array} \right\}$

$$T = \begin{array}{lllllll} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \\ 3 & 3 \end{array} \qquad \begin{array}{l} \text{shape}(T) = (7,3,3,3) \\ \text{type}(T) = (4,6,6) \end{array}$$



$T_{\le 1}$, $T_{\le 2}$, $T_{\le 3}$

Define

$$\chi_{\lambda/\mu} = (1-t)^{\#cc} \prod_{cc} (-t)^{r-1}, \qquad \chi_T = \prod_{i=1}^{\ell} \chi_{T_{\le i}/T_{<i}}$$

(first product over connected components,
r = # of rows in connected component)

The big Schur functions are

$$S_\lambda = \sum_{T \in BB(\lambda)} \chi_T x^T$$

Define P_μ by the following equation

$$J_\mu = h_\mu^{\mu} P_\mu = \sum_\lambda K_{\lambda\mu}(q,t) S_\lambda$$