30.06.2021
Workshop ①
Macdonald
A. Ram

Principal Specializations

Symmetric Macdonald polynomial

$$P_\lambda = P_\lambda(x;q,t) = P_\lambda(x_1,\ldots,x_n;q,t).$$

Schur function

$$s_\lambda = P_\lambda(x;t,t) = s_\lambda(x_1,\ldots,x_n) = \mathrm{char}(L(\lambda))$$

where $L(\lambda)$ is the irreducible integrable representation of $GL_n(\mathbb{C})$ of highest wt. λ.

$$\dim(L(\lambda)) = s_\lambda(1,\ldots,1)$$

$$= \prod_{b\in\lambda} \frac{n+c(b)}{h(b)}$$

$$\mathrm{qdim}(L(\lambda)) = s_\lambda(1,t,t^2,\ldots,t^{n-1})$$

$$= t^{n(\lambda)} \prod_{b\in\lambda} \frac{1-t^{n-c(b)}}{1-t^{h(b)}}$$

$$\mathrm{edim}(L(\lambda)) = P_\lambda(1,t,t^2,\ldots,t^{n-1};q,t)$$

$$= t^{n(\lambda)} \prod_{b\in\lambda} \frac{1-q^{\mathrm{coarm}_\lambda(b)}t^{n-\mathrm{coleg}_\lambda(b)}}{1-q^{\mathrm{arm}_\lambda(b)}t^{\mathrm{leg}_\lambda(b)+1}}$$

dimension

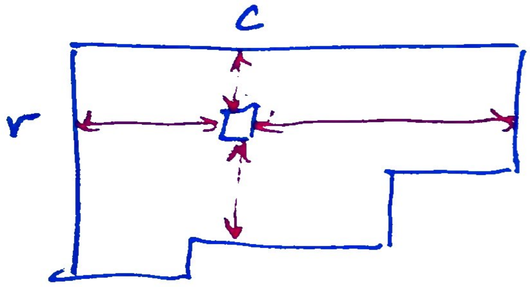
$\lim_{t\to 1}$

quantum dimension

$q=t$

elliptic dimension

Statistics



$\mathrm{coleg}_\lambda(b)=r-1$

$\mathrm{coarm}_\lambda(b)=c-1 \qquad \mathrm{arm}_\lambda(b)=\lambda_r-c$

$\mathrm{leg}_\lambda(b)=\lambda'_c-r$

<u>content of b</u> $\quad c(b)=c-r=\mathrm{coarm}_\lambda(b)-\mathrm{coleg}_\lambda(b)$

<u>hooklength of b</u> $\quad h(b)=\mathrm{arm}_\lambda(b)+\mathrm{leg}_\lambda(b)+1.$

$$n(\lambda)=\sum_{i=1}^{n}(i-1)\lambda_i$$

<u>General theorem</u> (Principal specializations)

$$ev_{-k\rho^\vee}(E_\mu)=t^{-\frac{1}{2}\ell(v_\mu^{-1})}\,ev_{k\rho}(c_{u_\mu}(Y^{-1}))$$

$$ev_{-k\rho^\vee}(P_\lambda)=ev_{-k\rho}(c_{t_\lambda}(Y^{-1}))$$

<u>For GL_n:</u> $-\rho^\vee=\left(-\left(\frac{n-1}{2}\right),-\left(\frac{n-3}{2}\right),\dots\left(\frac{n-3}{2}\right),\left(\frac{n-1}{2}\right)\right)$

$$=-\left(\frac{n-1}{2}\right)(1,1,\dots,1)+(0,1,2,\dots,n-1).$$

So $ev_{k\rho^\vee}(E_\mu)=t^{\left(\frac{n-1}{2}\right)|\mu|}E_\mu(t^0,t^1,t^2,\dots,t^{n-1})$

$$ev_{k\rho^\vee}(P_\lambda)=t^{-\left(\frac{n-1}{2}\right)|\lambda|}P_\lambda(t^0,t^1,t^2,\dots,t^{n-1})$$

where $|\mu|=\mu_1+\mu_2+\dots+\mu_n$.

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Roots, Inversions, and c-functions Workshop ③

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$$W = \{\text{bijections } w: \mathbb{Z} \to \mathbb{Z} \mid w(i+n) = w(i)+n\}.$$

$$\cup|$$

$$W_0 = S_n = \{\text{bijections } w: \{1,\dots,n\} \to \{1,\dots,n\}\}$$

$$\mathrm{Inv}(w) = \left\{(i,k) \;\middle|\; \begin{matrix} i \in \{1,\dots,n\},\ k \in \mathbb{Z} \\ i<k \text{ and } w(i) > w(k) \end{matrix}\right\}$$

$$\ell(w) = \#\mathrm{Inv}(w)$$

Alternative notation for roots:

$$\beta^\vee = (i, j+\ell n) = \varepsilon_i^\vee - \varepsilon_j^\vee + \ell K = \ell K + \alpha_{ij}$$

Define

$$c_{\beta^\vee} = c_{\beta^\vee}(Y) = \frac{t^{-\frac12} - t^{\frac12} Y^{\beta^\vee}}{1 - Y^{\beta^\vee}} = \frac{t^{-\frac12} - t^{\frac12} q^{-\ell} Y_i Y_j^{-1}}{1 - q^{-\ell} Y_i Y_j^{-1}}$$

since $Y^{-K} = q$, $Y_i = Y^{\varepsilon_i^\vee}$, $Y_j = Y^{\varepsilon_j^\vee}$.

Define

$$c_w = \prod_{\beta^\vee \in \mathrm{Inv}(w)} c_{\beta^\vee}$$

Define

$$ev_{-k\rho}\left(Y^{-(\ell K + \alpha_{ji})}\right) = q^\ell t^{j-i}$$

t_μ, v_μ and u_μ

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$. Define $t_\mu \in W$ by

$$t_\mu(1) = 1 + n\mu_1, \quad \ldots, \quad t_\mu(n) = n + n\mu_n.$$

Define $u_\mu \in W$ and $v_\mu \in W_0$ by

u_μ is min. length in $W_0 t_\mu W_0$

v_μ is min. length with $v_\mu \mu$ weakly inc.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

$$\mathrm{Inv}(t_\lambda) = \left\{ (i, j + \ell n) \;\middle|\; \begin{array}{l} i,j \in \{1, \ldots, n\} \\ i < j \text{ and } \ell \in \{0, 1, \ldots, \lambda_i - \lambda_j - 1\} \end{array} \right\}$$

$$= \{ \ell K + \alpha_{ij} \mid i < j \text{ and } 0 \leq \ell \leq \lambda_i - \lambda_j - 1 \}.$$

<u>Example</u> $\lambda = (5, 4, 4, 3, 2)$

		$2K + \alpha_{15}$	$K + \alpha_{15}$ $K + \alpha_{14}$	α_{15} α_{14} α_{13} α_{12}
		$K + \alpha_{25}$	α_{25} α_{24}	
		$K + \alpha_{35}$	α_{35} α_{34}	
		α_{45}		

Start with

$$ev_{-\kappa\rho^\vee}(P_\lambda) = ev_{-\kappa\rho}\left(c_{t_\lambda}(Y^{-1})\right)$$

$$= ev_{-\kappa\rho}\left(\prod_{\beta^\vee \in Inv(t_\lambda)} c_{\beta^\vee}(Y^{-1})\right)$$

Consider the box $b=(1,4)$.

$$ev_{-\kappa\rho}\left(c_{K+\alpha_{15}}(Y^{-1})\, c_{K+\alpha_{14}}(Y^{-1})\right)$$

$$= ev_{-\kappa\rho}\left(\frac{(t^{-\frac12} - t^{\frac12} Y^{-(K+\alpha_{15})})}{(1-Y^{-(K+\alpha_{15})})}\,\frac{(t^{-\frac12} - t^{\frac12} Y^{-(K+\alpha_{14})})}{(1-Y^{-(K+\alpha_{14})})}\right)$$

$$= t^{-\frac12\cdot 2}\, ev_{-\kappa\rho}\left(\frac{(1-tqY^{-\alpha_{15}})}{(1-qY^{-\alpha_{15}})}\,\frac{(1-tqY^{-\alpha_{14}})}{(1-qY^{-\alpha_{14}})}\right)$$

$$= t^{-\frac12\cdot 2}\,\frac{(1-tq\,t^{5-1})}{\cancel{(1-qt^{5-1})}}\cdot\frac{\cancel{(1-tqt^{4-1})}}{(1-qt^{4-1})}$$

$$= t^{-\frac12\cdot 2}\,\frac{1-tqt^{5-1}}{1-qt^{4-1}}$$

So $\boxed{\begin{matrix} K+\alpha_{15} \\ K+\alpha_{14} \end{matrix}}$ gives $\boxed{t^{-\frac12\cdot 2}\,\frac{1-qt^{5-0}}{1-qt^{4-1}}}$

So

$\mathrm{Inv}(t_\lambda) =$

		$2K+\alpha_{15}$	$K+\alpha_{15}$ $K+\alpha_{14}$	α_{15} α_{14} α_{13} α_{12}
		$K+\alpha_{25}$	α_{25} α_{24}	
		$K+\alpha_{35}$	α_{35} α_{34}	
		α_{45}		

gives

$t^{\sum \ell(t_\lambda)} ev_{-k\rho}(c_{t_\lambda}(Y^{-1})) =$

		$\frac{1-q^2t^{5-0}}{1-q^2t^{5-1}}$	$\frac{1-qt^{5-0}}{1-q^1t^{4-1}}$	$\frac{1-q^0t^{5-0}}{1-q^0t^{2-1}}$
		$\frac{1-q^1t^{5-1}}{1-q^1t^{5-2}}$	$\frac{1-q^0t^{5-1}}{1-q^0t^{4-2}}$	
		$\frac{1-q^1t^{5-2}}{1-q^1t^{5-3}}$	$\frac{1-q^0t^{5-2}}{1-q^0t^{4-3}}$	
		$\frac{1-q^0t^{5-3}}{1-q^0t^{5-4}}$		

which is equal to

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$\frac{1-q^0t^{5-0}}{1}$	$\frac{1-q^1t^{5-0}}{1}$	$\frac{1-q^2t^{5-0}}{1-q^2t^{5-1}}$	$\frac{1}{1-q^1t^{4-1}}$	$\frac{1}{1-q^0t^{2-1}}$
$\frac{1-q^0t^{5-1}}{1}$	$\frac{1-q^1t^{5-1}}{1}$	$\frac{1}{1-q^1t^{5-2}}$	$\frac{1}{1-q^0t^{4-2}}$	
$\frac{1-q^0t^{5-2}}{1}$	$\frac{1-q^1t^{5-2}}{1}$	$\frac{1}{1-q^1t^{5-3}}$	$\frac{1}{1-q^0t^{4-3}}$	
$\frac{1-q^0t^{5-3}}{1}$		$\frac{1}{1-q^0t^{5-4}}$		

=

$\frac{1-q^0t^{5-0}}{1-q^4t^{4+1}}$	$\frac{1-q^1t^{5-0}}{1-q^3t^{4+1}}$	$\frac{1-q^2t^{5-0}}{1-q^2t^{5-1}}$	$\frac{1-q^3t^{5-0}}{1-q^1t^{4-1}}$	$\frac{1-q^4t^{5-0}}{1-q^0t^{2-1}}$
$\frac{1-q^0t^{5-1}}{1-q^3t^{3+1}}$	$\frac{1-q^1t^{5-1}}{1-q^2t^{3+1}}$	$\frac{1-q^2t^{5-1}}{1-q^1t^{5-2}}$	$\frac{1-q^3t^{5-1}}{1-q^0t^{4-2}}$	
$\frac{1-q^0t^{5-2}}{1-q^3t^{2+1}}$	$\frac{1-q^1t^{5-1}}{1-q^2t^{2+1}}$	$\frac{1-q^2t^{5-2}}{1-q^1t^{5-3}}$	$\frac{1-q^3t^{5-2}}{1-q^0t^{4-3}}$	
$\frac{1-q^0t^{5-3}}{1-q^2t^{1+1}}$	$\frac{1-q^1t^{5-3}}{1-q^1t^{1+1}}$	$\frac{1-q^2t^{5-3}}{1-q^0t^{5-4}}$		
$\frac{1-q^0t^{5-4}}{1-q^1t^{0+1}}$	$\frac{1-q^1t^{5-4}}{1-q^0t^{0+1}}$			

$$= \prod_{b\in\lambda} \frac{1-q^{\mathrm{coarm}_\lambda(b)}t^{n-\mathrm{coleg}_\lambda(b)}}{1-q^{\mathrm{arm}_\lambda(b)}t^{\mathrm{leg}_\lambda(b)+1}}.$$