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Working Seminar
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(1)

The quantum group is a fake

The enveloping algebra for $\mathfrak{sl}_2$

U is generated by e, f, H and the finite dimensional irreducible representations of U are

$$\pi_{\ell\omega_1} : U \to M_{\ell+1}(\mathbb{C}) \quad \text{for } \ell \in \mathbb{Z}_{\geq 0}$$

given by

$$\pi_{\ell\omega_1}(e) = \begin{pmatrix} 0 & \ell & & & \\ & 0 & \ell-1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 2 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \qquad \pi_{\ell\omega_1}(f) = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 2 & \ddots & & & \\ & & \ddots & 0 & & \\ & & & \ell-1 & 0 & \\ & & & & \ell & 0 \end{pmatrix}$$

$$\pi_{\ell\omega_1}(H) = \begin{pmatrix} \ell & & & & \\ & \ell-2 & & & \\ & & \ddots & & \\ & & & -(\ell-2) & \\ & & & & -\ell \end{pmatrix}$$

Alternatively, U is generated by e, f, H with relations

$$ef = fe + H, \quad eH = (H-2)e, \quad Hf = f(H-2)$$

Coproduct: Define a homomorphism $\Delta : U \to U \otimes U$ by

$$\Delta(e) = e \otimes 1 + 1 \otimes e, \quad \Delta(f) = f \otimes 1 + 1 \otimes f, \quad \Delta(H) = H \otimes 1 + 1 \otimes H$$

Associator: Define $\Phi \in U \otimes U \otimes U$ by $\Phi = 1 \otimes 1 \otimes 1$.

R-matrix: Define $R \in U \otimes U$ by $R = 1 \otimes 1$.

The quantum group U_q for sl_2

U_q is generated by X^+, X^-, H and the finite dimensional irreducible representations of U_q are

$$\pi_{\ell\omega_1} : U_q \to M_{\ell+1}(\mathbb{C}) \quad \text{given by}$$

$$\pi_{\ell\omega_1}(X^+) = \begin{pmatrix} 0 & [\ell] & & & \\ & 0 & [\ell-1] & & \\ & & \ddots & \ddots & \\ & & & 0 & [2] & \\ & & & & 0 & [1] \\ & & & & & 0 \end{pmatrix} \qquad \pi_{\ell\omega_1}(X^-) = \begin{pmatrix} 0 & & & & & \\ [1] & 0 & & & & \\ & [2] & 0 & & & \\ & & \ddots & \ddots & & \\ & & & [\ell-1] & 0 & \\ & & & & [\ell] & 0 \end{pmatrix}$$

$$\pi_{\ell\omega_1}(H) = \begin{pmatrix} \ell & & & & \\ & \ell-2 & & & \\ & & \ddots & & \\ & & & -(\ell-2) & \\ & & & & -\ell \end{pmatrix} \quad \text{with} \quad q = e^{\hbar/2}, \quad [k] = \left(\frac{q^k - q^{-k}}{q - q^{-1}}\right)\left(\frac{q - q^{-1}}{2 \cdot \hbar/2}\right)$$

Alternatively, U_q is generated by X^+, X^-, H with relations

$$X^-X^+ = X^+X^- - \left(\frac{K - K^{-1}}{q - q^{-1}}\right)\left(\frac{q - q^{-1}}{2 \cdot \hbar/2}\right), \quad X^+H = (H-2)X^+, \quad HX^- = X^-(H-2)$$

Coproduct: Define a homomorphism $\Delta_q : U_q \to U_q \otimes U_q$ by

$$\Delta_q(X^+) = X^+ \otimes e^{\frac{1}{4}\hbar H} + e^{-\frac{1}{4}\hbar H} \otimes X^+$$

$$\Delta_q(X^-) = X^- \otimes e^{\frac{1}{4}\hbar H} + e^{-\frac{1}{4}\hbar H} \otimes X^- \qquad \Delta_q(H) = H \otimes 1 + 1 \otimes H$$

Associator: Define $\Phi_q \in U_q \otimes U_q \otimes U_q$ by $\Phi_q = 1 \otimes 1 \otimes 1$

R-matrix: Define $R_q \in U_q \otimes U_q$ by

$$R_q = e^{\frac{1}{4}\hbar(H \otimes H + 1 \otimes H + H \otimes 1)} \sum_{k \in \mathbb{Z}_{\geq 0}} \hbar^k e^{k\frac{\hbar}{4}} \left(\prod_{i=1}^{k} \frac{q^2 - 1}{q^{2k} - 1}\right) (X^+)^k \otimes (X^-)^k$$

The nonquantum group for $\mathfrak{sl}_2$

$\mathcal{U}$ is generated by e, f, H and the finite dimensional irreducible representations of $\mathcal{U}$ are

$$\pi_{\ell\omega_1} : \mathcal{U} \to M_{\ell+1}(\mathbb{C}) \text{ given by}$$

$$\pi_{\ell\omega_1}(e) = \begin{pmatrix} 0 & \ell & & & & \\ & 0 & \ell-1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 2 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \qquad \pi_{\ell\omega_1}(f) = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 2 & \ddots & & & \\ & & \ddots & 0 & & \\ & & & \ell-1 & 0 & \\ & & & & \ell & 0 \end{pmatrix}$$

$$\pi_{\ell\omega_1}(H) = \begin{pmatrix} \ell & & & & \\ & \ell-2 & & & \\ & & \ddots & & \\ & & & -(\ell-2) & \\ & & & & -\ell \end{pmatrix}$$

Alternatively $\mathcal{U}$ is generated by e, f, H with relations

$$fe = ef - H, \qquad eH = (H-2)e, \qquad Hf = f(H-2)$$

Coproduct Define a homomorphism $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$ by

$$\Delta(e) = e\otimes 1 + 1\otimes e, \quad \Delta(f) = f\otimes 1 + 1\otimes f, \quad \Delta(H) = H\otimes 1 + 1\otimes H$$

Coassociator Define $\Phi_h \in \mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}$ by $\Phi_h = e^{\frac{1}{4}h[t^{12}, t^{13}]}$

where

$$[t^{12}, t^{23}] = e\otimes f\otimes H - e\otimes H\otimes f - f\otimes e\otimes H + H\otimes e\otimes f + f\otimes H\otimes e - H\otimes f\otimes e$$

R-matrix Define $R_h \in \mathcal{U}\otimes\mathcal{U}$ by $R_h = e^{h/2\, t}$ where

$$t = e\otimes f + f\otimes e + \tfrac{1}{2}(H\otimes H)$$

$$(\mathcal{U}, \Delta, \Phi_h, R_h) = (\mathcal{U}, \Delta, e^{\frac{1}{4}h[t^{12}, t^{23}]}, e^{h/2\, t})$$

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The quantum group is a fake

Theorem

(a) Define

$$Q = \left(\frac{\sinh(\frac{h}{2})}{h/2}\right)^{-1} \sum_{k \in \mathbb{Z}_{\ge 0}} \frac{(\frac{h}{2})^{2k}}{(2k)!} \left(\sum_{j=0}^{k-1} ((H+1)^2 + 4fe)^{k-1-j} ((H+1)^2)^j \right)$$

Then the homomorphism $\gamma : U_q \to U$ given by

$$\gamma(H) = H, \quad \gamma(X^+) = e, \quad \gamma(X^-) = f \cdot Q$$

is an algebra isomorphism.

(b) Define $\tilde{\Delta} : U \to U \otimes U$ and $\tilde{R} \in U \otimes U$ by

$$\tilde{\Delta} = (\gamma \otimes \gamma) \circ \Delta \circ \gamma^{-1} \quad \text{and} \quad \tilde{R} = (\gamma \otimes \gamma)(R_q).$$

Define

$$F = 1 \otimes 1 + h F_1 + h^2 F_2 + \cdots$$

Then

$$F^{-1} \tilde{\Delta}(a) F = \Delta(a) \quad \text{for } a \in U$$

$$(id \otimes \Delta)(F^{-1}) (F^{23})^{-1} F^{12} (\Delta \otimes id)(F) = \Phi_h$$

$$(F^{21})^{-1} \tilde{R} F = e^{h/2 t}$$

Summary

$$(U_q, \Delta_q, 1 \otimes 1 \otimes 1, R_q) \xrightarrow{\gamma} (U, \tilde{\Delta}, 1 \otimes 1 \otimes 1, \tilde{R}) \xrightarrow{F} (U, \Delta, \Phi_h, e^{\frac{h}{2} t}).$$