12.10.2021
standard
A. Ram

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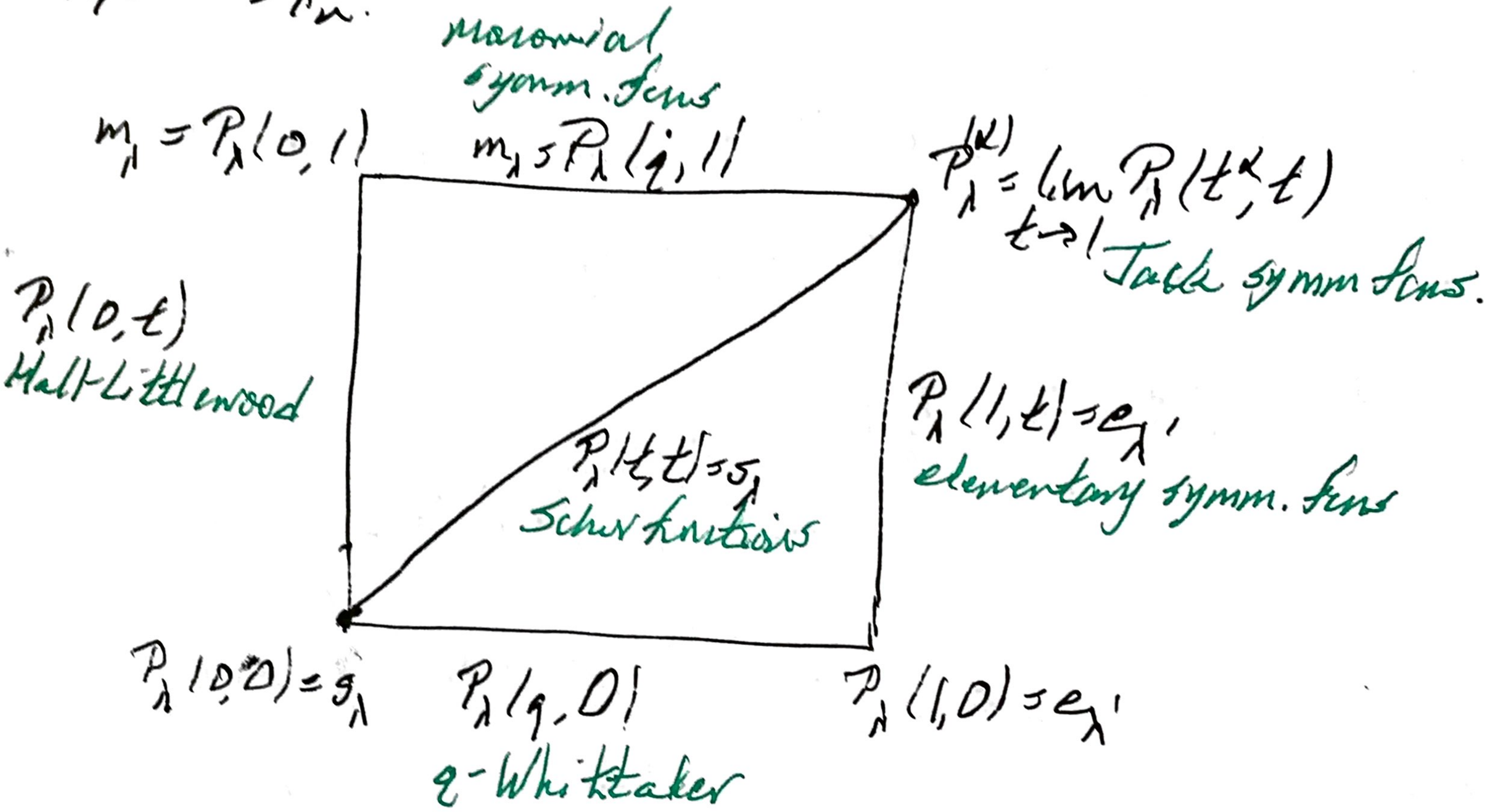
Symmetric Macdonald polynomials

$P_\lambda(q,t) = P_\lambda(x;q,t) = P_\lambda(x_1,\ldots,x_n;q,t)$ are indexed by partitions $\lambda = (\lambda_1,\ldots,\lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n$.



Example $P_{(2,1,0)}$

$$= m_{(2,1,0)} + \left(\frac{(1-t^2)(1-q^2t)}{(1-qt)(1-qt^2)} + \frac{(1-t)(1-q^2)}{(1-q)(1-qt)}\right) m_{(1,1,1)}$$

$$= m_{(2,1,0)} + \left(\frac{(1-t^{-2})(1-q^{-2}t^{-1})}{(1-q^{-1}t^{-1})(1-q^{-1}t^{-2})} + \frac{(1-t^{-1})(1-q^{-2})}{(1-q^{-1})(1-q^{-1}t^{-1})}\right) m_{(1,1,1)}$$

$$P_\lambda(q,t) = P_\lambda(q^{-1},t^{-1})$$

$P_\lambda(t,t)$ does not depend on t

Nonsymmetric Macdonald polynomials

$E_\mu(q,t) = E_\mu(x;q,t) = E_\mu(x_1,\ldots,x_n;q,t)$ are indexed by $\mu = (\mu_1,\ldots,\mu_n) \in \mathbb{Z}^n$. Since

$$E_\mu(q,t) = x^\mu + \sum_{\nu<\mu} a_{\mu\nu}(q,t)x^\nu$$

where $x^\mu = x_1^{\mu_1}\cdots x_n^{\mu_n}$ then

$\{E_\mu \mid \mu \in \mathbb{Z}^n\}$ form a basis of $\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$.

A nice formula for $P_\lambda(q,t)$

$$P_\lambda(q,t) = \frac{1}{W_\lambda(t)} \sum_{w\in S_n} w\Big(E_\lambda \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\Big)$$

Since $E_\lambda(0,t) = x^\lambda$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ then

$$P_\lambda(0,t) = \frac{1}{W_\lambda(t)} \sum_{w\in S_n} w\Big(x^\lambda \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\Big)$$

$$s_\lambda = P_\lambda(0,0) = \sum_{w\in S_n} w\Big(x^\lambda \prod_{i<j} \frac{x_i}{x_i - x_j}\Big)$$

$$m_\lambda = P_\lambda(0,1) = \sum_{w\in S_n} w\Big(x^\lambda \prod_{i<j} \frac{x_i - x_j}{x_i - x_j}\Big).$$

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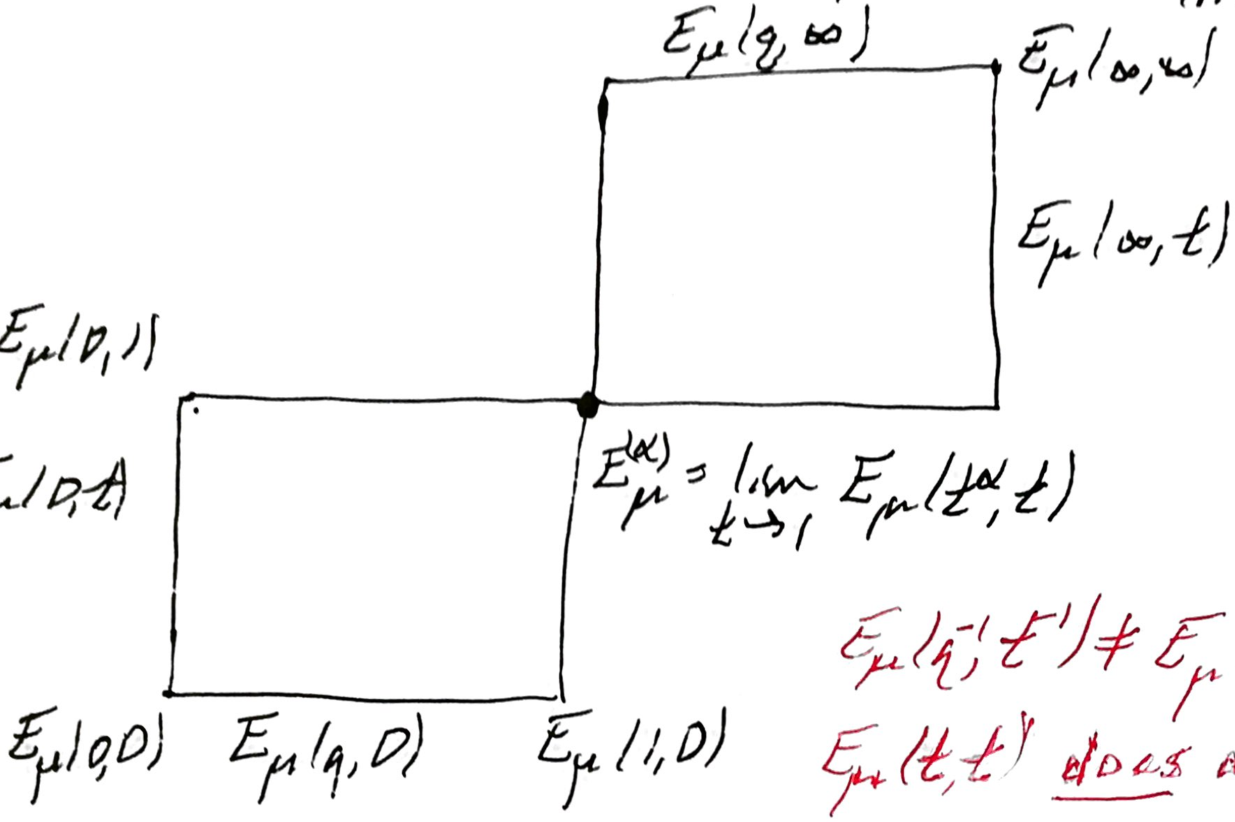
Specialization square for E_μ



$E_\mu(q^{-1},t^{-1}) \neq E_\mu(q,t)$

$E_\mu(t,t)$ does depend on t

Example

$$E_{(1,0,\ldots,0)}(q,t) = x_i + \frac{1-t}{1-qt^{n-(i-1)}}(x_{i-1}+\cdots+x_1)$$

$$= x_i + \frac{(1-t^{-1})q^{-1}t^{(n-(i-2))}}{1-q^{-1}t^{-(n-(i-1))}}(x_{i-1}+\cdots+x_1)$$

$$E_{(1,0,\ldots,0)}(t,t) = x_i + \frac{1-t}{1-t^{n-(i-2)}}(x_{i-1}+\cdots+x_1)$$

$$E^{(\alpha)}_{(1,0,\ldots,0)} = x_i + \frac{1}{\alpha+n-(i-1)}(x_{i-1}+\cdots+x_1).$$

Monomial expansion formulas

$$E_\mu(q,t) = \sum_T q^{\mathrm{maj}_-(T)} t^{\mathrm{cov}_-(T)} \left(\prod_{k \in T} \frac{1-t}{1-q^{\mathrm{sh}(k)} t^{\mathrm{ht}(k)}} \right) x^T$$

$$= \sum_T q^{-\mathrm{maj}_+(T)} t^{-\mathrm{cov}_+(T)} \left(\prod_{k \in T} \frac{1-t^{-1}}{1-q^{-\mathrm{sh}(k)} t^{\mathrm{ht}(k)}} \right) x^T$$

with $\mathrm{maj}_\pm(T) \in \mathbb{Z}_{\geq 0}$, $\mathrm{cov}_\pm(T) \in \mathbb{Z}_{\geq 0}$ and $\mathrm{sh}(k) \in \mathbb{Z}_{>0}$, $\mathrm{ht}(k) \in \mathbb{Z}_{>0}$.

These give

$$E_\lambda(0,t) = x^\lambda \quad \text{if } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$$E_\lambda(\infty,t) = x^\lambda \quad \text{if } \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

E-expansion formula

$$P_\lambda(q,t) = \sum_{\mu \in S_n \lambda} \left(\prod_{\substack{i<j \\ \mu_i > \mu_j}} t \frac{1-q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j) - 1}}{1-q^{\mu_i-\mu_j} t^{v_\mu(i) - v_\mu(j)}} \right) E_\mu(q,t)$$

where $v_\mu \in S_n$ is min. length such that $v_\mu \mu$ is weakly increasing.

So $P_\lambda(q,0) = E_{w_0\lambda}(q,0)$

where $w_0\lambda = (\lambda_n, \ldots, \lambda_1)$ if $\lambda = (\lambda_1, \ldots, \lambda_n)$

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Creation formula for E_μ

(a) $E_{(0,\ldots,0)} = 1$

(b) $E_{(\mu_n+1,\mu_1,\ldots,\mu_{n-1})} = q^{-\mu_n} x_1 E_{(\mu_1,\ldots,\mu_n)}(x_2,\ldots,x_n,q^{-1}x_1)$

(c) If $\mu_i > \mu_{i+1}$ then

$$E_{s_i\mu} = \left(\partial_i x_i - t x_i \partial_i + \frac{(1-t)q^{m}t^{\xi}}{1-q^{m}t^{\xi}}\right) E_\mu$$

where $\partial_i f = \dfrac{f - s_i f}{x_i - x_{i+1}}$ and $q^{m}t^{\xi} = q^{\mu_i-\mu_{i+1}} t^{v_\mu(i)-v_\mu(i+1)}$

with $v_\mu \in S_n$ min. length such that $v_\mu\mu$ is weakly increasing.

i.e.

$$E_\mu = \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \cdot 1 \qquad \underline{\text{Creation formula}}$$

This might remind you of:

(a) <u>Demazure character formula</u>

$$\mathrm{char}(L(\lambda)_{\le w}) = D_{i_1}\cdots D_{i_\ell} x^\lambda \qquad GL_n(\mathbb{C}((\epsilon)))$$

(b) <u>Formula for Iwahori-spherical functions</u>

$$IwIx^\lambda K = T_{i_1}\cdots T_{i_\ell} x^\lambda \qquad GL_n(\mathbb{Q}_p)$$

where $w = s_{i_1}\cdots s_{i_\ell}$ is a reduced word

Ion's theorems

(a) $$x^{-\lambda_0} D_i x^{\lambda_0} E_\mu = \left(y^{-\delta_{i0}\alpha_0^\vee} t^{\frac12} T_i^\vee + t\, x_i \partial_i - \frac{(1-t)q^m t^j}{1-q^m t^j} \right) E_\mu$$

with $m, j \in \mathbb{Z}_{>0}$. So at $t=0$

$$x^{-\lambda_0} D_i x^{\lambda_0} = y^{-\delta_{i0}\alpha_0^\vee} t^{\frac12} T_i^\vee \quad \text{and}$$

$E_\mu(q,0)$ is a level 1 Demazure character

(b) $$t^{\frac12} T_i^{-1} E_\mu = \left(t^{\frac12} T_i^\vee - \frac{(1-t)q^m t^j}{1-q^m t^j} \right) E_\mu$$

with $m, j \in \mathbb{Z}_{>0}$. So at $q=0$

$$t^{\frac12} T_i^{-1} = t^{\frac12} T_i^\vee \quad \text{and}$$

$E_\mu(0,t)$ is an Iwahori spherical function.