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Transvections and Hecke algebras MATRIX ①

A. Ram.

$G = GL_n(\mathbb{F}_q)$
$\cup\!|$
$B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$

Bruhat decomposition

$$G = \bigsqcup_{w \in S_n} BwB$$

$\mathcal{C} = \mathcal{C}_{(21^{n-2})} = \{\text{transvections}\}$ is the conjugacy class of $u_{(21^{n-2})} = \begin{pmatrix} 1 & 1 & & 0 \\ 0 & 1 & & \\ & & 1 & \\ 0 & & & \ddots \ 1 \end{pmatrix}$

Lattice of subspaces in $\mathbb{F}_q^n$

$\mathcal{L} = \{\text{subspaces } V \subseteq \mathbb{F}_q^n\}$ ordered by inclusion ranked by $\dim(V)$

$$\mathcal{F}l_n = \left\{ \begin{matrix}\text{maximal chains} \\ \text{in } \mathcal{L}\end{matrix} \right\} \xrightarrow{\sim} G/B$$

$$(0 \subseteq V_1 \subseteq \cdots \subseteq V_n) \longleftarrow gB$$

where $V_i = \text{span}\{\text{first } i \text{ columns of } g\}$.

The Walk: From F.
- choose a random transvection t
- move from F to tF
- report the double coset of tF

The viewer sees a walk on S_n

The permutation representation of G on Fl_n is

$$\mathbb{1}_B^G = \mathrm{Ind}_B^G(\mathrm{triv}).$$

The <u>Hecke algebra</u> is

$$H_n = \mathrm{End}_G(\mathbb{1}_B^G).$$

A basis $\{T_w \mid w \in S_n\}$ of H_n is given by

$$T_w = \frac{1}{\mathrm{Card}(B)} \sum_{x \in BwB} x$$

(acting on $\mathbb{1}_B^G$ on the right) and

$$T_{s_i} T_w = \begin{cases} T_{s_i w}, & \text{if } \ell(s_i w) > \ell(w), \\ q T_{s_i w} + (q-1) T_w, & \text{if } \ell(s_i w) < \ell(w), \end{cases}$$

where

$s_i = (i, i+1)$ is a transposition

$\ell(w) = \#$ of inversions of w.

Let

$$C = \sum_{t \in \mathcal{C}} t \qquad \text{in } Z(\mathbb{C}[G]).$$

C acts on $\mathbb{1}_B^G$ and on H_n.

Theorem (Diaconis-R-Simper)

C acts on $\mathcal{L}_B^G$ the same way as

$$D = (n-1)q^{n-1} - \left(\frac{q^{n-1}-1}{q-1}\right) + (q-1)\sum_{i<j} q^{n-1-(j-i)} t_{(i,j)}$$

The eigenvalues of D are

$$e_\lambda = \left(q^{n-1}\sum_{b\in\lambda} q^{ct(b)}\right) - \left(\frac{q^n-1}{q-1}\right)$$

where $ct(b) = j-i$ if b is in position (i,j) of λ.

The multiplicity of e_λ on H_n is

$$f^\lambda = \frac{n!}{\prod_{b\in\lambda} h(b)}$$

where $h(b) = $ (hook length of b in λ).

The walk converges to stationarity in order $\frac{\log(n)}{\log(q)}$ steps