The KZ functor: Type $G(r,1,1)$

KZ lecture ①

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<u>Goal</u>: KZ functor in examples

$$KZ: \left\{\begin{matrix}\text{RCA}\\ \text{modules}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Hecke}\\ \text{modules}\end{matrix}\right\}$$

Type $G(r,1,1)$ example

Let $r \in \mathbb{Z}_{>0}$ and $\zeta = e^{2\pi i/r}$.

The Hecke algebra H_{fin}

Let $q_0, q_1, \ldots, q_{r-1} \in \mathbb{C}$.

The <u>Hecke algebra</u> H_{fin} is the algebra generated by T_1 with relation

$$(T_1 - q_0)(T_1 - q_1)\cdots(T_1 - q_{r-1}) = 0$$

The rational Cherednik algebra

Let $\kappa \in \mathbb{C}$ and $c_1, \ldots, c_{r-1} \in \mathbb{C}$.

The <u>rational Cherednik algebra</u> is the algebra $\mathbb{H}$ generated by x_1, y_1, t_1 with relations

$$t_1^r = 1, \quad t_1 x_1 = \zeta x_1 t_1, \quad t_1 y_1 = \zeta y_1 t_1$$

$$y_1 x_1 = x_1 y_1 + \kappa - \sum_{\ell=1}^{r-1} c_\ell (1 - \zeta^{-\ell}) t_1^\ell$$

(see [GGOR §3.1] and [Gri, Prop 4.1 Eqn (4.9)]).

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Modules

Let $\mathbb{C}[W]$ be the algebra generated by t_1 with relation

$$t_1^r = 1.$$

Let E be the $\mathbb{C}[W]$-module

$$E = \mathrm{span}\{e_0, \ldots, e_{r-1}\} \quad \text{with} \quad t_1 e_j = \zeta^j e_j$$

for $j \in \{0, 1, \ldots, r-1\}$. In other words, in the basis $\{e_0, \ldots, e_{r-1}\}$, t_1 acts by the matrix

$$t_1 = \begin{pmatrix} 1 & & & 0 \\ & \zeta & & \\ & & \zeta^2 & \\ & & & \ddots & \\ 0 & & & & \zeta^{r-1} \end{pmatrix}$$

The $\mathbb{H}$-module $\Delta(E)$

Let $\Delta(E)$ be the $\mathbb{H}$-module generated by $e_0, e_1, \ldots, e_{r-1}$ with

$$t_1 e_j = \zeta^j e_j \quad \text{and} \quad y_1 e_j = 0.$$

Then

$$\Delta(E) \text{ has basis } \left\{ x_1^a e_j \;\middle|\; \begin{matrix} a \in \mathbb{Z}_{\geq 0} \\ j \in \{0, 1, \ldots, r-1\} \end{matrix} \right\}$$

The H_{fin}-module $KZ(\Delta(E))$

Let $KZ(\Delta(E))$ be the H_{fin}-module

$KZ(\Delta(E)) = \mathrm{span}\{e_0, e_1, \ldots, e_{r-1}\}$ with

$$T_1 e_j = q_j e_j, \qquad \text{for } j \in \{0, 1, \ldots, r-1\}.$$

In other words, in the basis $\{e_0, e_1, \ldots, e_{r-1}\}$ T_1 acts by the matrix

$$T_1 = \begin{pmatrix} q_0 & & & \\ & q_1 & & \\ & & \ddots & \\ & & & q_{r-1} \end{pmatrix}$$

<u>The point</u>:

$$\begin{array}{rcl} KZ : \{\mathbb{H}\text{-modules}\} & \longrightarrow & \{H_{fin}\text{ modules}\} \\ \Delta(E) & \longmapsto & KZ(\Delta(E)) \end{array}$$

If $\mathbb{H}$ has parameters ~~$q_0, \ldots, q_{r-1}$~~ $k, c_1, \ldots, c_{r-1}$ then H_{fin} <u>must have parameters</u> $q_0, \ldots, q_{r-1}$ given by

$$q_j = \zeta^{-j} e^{2\pi i k_j}, \qquad \text{where } k_j = r \sum_{\ell=1}^{r-1} \zeta^{\ell j} c_\ell$$

(see [GGOR §5.15] and [Gri, eqns (4.10) and (4.12)])

Polynomials and $D(V) \rtimes \mathbb{C}W$

Let $V = \mathbb{C}^1$.

Let $D(V) \rtimes \mathbb{C}W$ be the algebra generated by x_1, ∂_1 and t_1 with relations

$$t_1 x_1 = \zeta x_1 t_1, \quad t_1 \partial_1 = \zeta \partial_1 t_1, \quad \partial_1 x_1 = x_1 \partial_1 + 1.$$

Let $\mathbb{C}[V]$ be the $(D(V) \rtimes W)$-module generated by $\mathbb{1}$ with

$$t_1 \mathbb{1} = \mathbb{1} \quad \text{and} \quad \partial_1 \mathbb{1} = 0.$$

Then $\mathbb{C}[V]$ has basis $\{x_1^a \mathbb{1} \mid a \in \mathbb{Z}_{\geq 0}\}$ and

$$t_1 x_1^a \mathbb{1} = \zeta^a x_1^a \mathbb{1}$$

$$x_1 x_1^a \mathbb{1} = x_1^{a+1} \mathbb{1}$$

$$\partial_1 x_1^a \mathbb{1} = a x_1^{a-1} \mathbb{1}$$

The last identity is proved by induction on a,

$$\partial_1 x_1^a \mathbb{1} = \partial_1 x_1 x_1^{a-1} \mathbb{1} = (x_1 \partial_1 + 1) x_1^{a-1} \mathbb{1}$$

$$= (x_1 (a-1) x_1^{a-2} + x_1^{a-1}) \mathbb{1} = a x_1^{a-1} \mathbb{1}.$$

So, if $p = a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_\ell x_1^\ell \in \mathbb{C}[x_1]$ then

$$\partial_1 p \mathbb{1} = \left(\frac{\partial}{\partial x_1} p\right) \mathbb{1}.$$

Polynomials and $\tilde{H}$

Let $\Delta(\mathbb{1})$ be the $\tilde{H}$-module generated by $\mathbb{1}$ with

$$t_1\mathbb{1} = \mathbb{1} \quad\text{and}\quad y_1\mathbb{1} = 0.$$

Then $\Delta(\mathbb{1})$ has basis $\{x_1^a\mathbb{1} \mid a \in \mathbb{Z}_{\geq 0}\}$ and

$$t_1 x_1^a \mathbb{1} = \zeta^a x_1^a \mathbb{1},$$

$$x_1 x_1^a \mathbb{1} = x_1^{a+1}\mathbb{1}$$

$$y_1 x_1^a \mathbb{1} = \left(\kappa \frac{\partial}{\partial x_1} + \sum_{j=1}^{r-1} r k_j \frac{1}{x_1} z^{(j)}\right) x_1^a \mathbb{1}$$

where $z^{(j)} = \frac{1}{r}\sum_{l=0}^{r-1} \zeta^{jl} t_1^l$

(see [GGOR Remark 3.2] and [Gri; eqn(4.12)]).

The last identity is proved by induction on a.

Laurent polynomials and $\mathcal{D}(V^0)\rtimes W$

Let $V = \mathbb{C}^1$ and $V^0 = \mathbb{C}^1 - \{0\} = \mathbb{C}^\times$

Let $\mathcal{D}(V^0)\rtimes W$ be the algebra generated by $x_1, x_1^{-1}, \partial_1, t_1$ with relations

$$t_1 x_1 = \zeta x_1 t_1, \quad t_1\partial_1 = \zeta \partial_1 t_1, \quad x_1 x_1^{-1} = x_1^{-1}x_1 = 1$$

$$\partial_1 x_1 = x_1\partial_1 + 1.$$

Let $\mathbb{C}[V^0]$ be the $(\mathcal{D}(V^0)\rtimes W)$-module generated by $\mathbb{1}$ with

$$t_1\mathbb{1} = \mathbb{1} \quad \text{and} \quad \partial_1\mathbb{1} = 0.$$

Then $\mathbb{C}[V^0]$ has basis $\{x_1^a\mathbb{1} \mid a\in\mathbb{Z}\}$ and

$$t_1 x_1^a\mathbb{1} = \zeta^a x_1^a \mathbb{1},$$

$$x_1^{\pm 1}x_1^a\mathbb{1} = x_1^{a\pm 1}\mathbb{1},$$

$$\partial_1 x_1^a\mathbb{1} = a x_1^{a-1}\mathbb{1}.$$

If $p = a_{-m}x_1^{-m} + \cdots + a_{-1}x_1^{-1} + a_0 + a_1 x_1 + \cdots + a_\ell x_1^\ell \in \mathbb{C}[x_1, x_1^{-1}]$ then

$$\partial_1 p\mathbb{1} = \left(\frac{\partial}{\partial x_1}p\right)\mathbb{1}.$$

Adding x_1^{-1} to $\tilde{\mathbb{H}}$

Let $\tilde{\mathbb{H}}^0$ be the algebra generated by x_1, x_1^{-1}, y_1, t_1 with relations

$$t_1^r = 1, \quad t_1 x_1 = \zeta x_1 t_1, \quad t_1 y_1 = \zeta y_1 t_1, \quad x_1 x_1^{-1} = x_1^{-1} x_1 = 1$$

$$y_1 x_1 = x_1 y_1 + \kappa - \sum_{\ell=1}^{r-1} c_\ell (1-\zeta^{-\ell}) t_1^\ell .$$

Then, as operators on $\mathbb{C}[V^0]$

$\tilde{\mathbb{H}}^0$ is the same as $\mathcal{D}(V^0) \rtimes W$

where the conversion is given by

$$y_1 = \kappa \partial_1 + \sum_{j=1}^{r-1} r k_j \frac{1}{x_1} z^{(j)}$$

and

$$\partial_1 = \frac{1}{\kappa} y_1 - \frac{1}{\kappa} \sum_{j=1}^{r-1} r k_j \frac{1}{x_1} z^{(j)} .$$