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KZ

Conversion 1: $\mathbb{C}[W]$ to Z_3

Let $r \in \mathbb{Z}_{>0}$ and $\zeta = e^{2\pi i/r}$

$\mathbb{C}[W]$ is the algebra generated by
t_1 with $t_1^3 = 1$.

Z_3 is the algebra generated by $z^{(0)}, z^{(1)}, z^{(2)}$
with relations

$$(z^{(i)})^2 = z^{(i)} \quad \text{and} \quad z^{(i)} z^{(j)} = z^{(j)} z^{(i)} = 0.$$

In fact, $\mathbb{C}[W]$ and Z_3 are <u>the same</u>.

The conversion is

$$z^{(0)} = \tfrac{1}{3}(1 + t_1 + t_1^2) \qquad 1 = z^{(0)} + z^{(1)} + z^{(2)}$$

$$z^{(1)} = \tfrac{1}{3}(1 + \zeta^2 t_1 + \zeta t_1^2) \qquad t_1 = z^{(0)} + \zeta z^{(1)} + \zeta^2 z^{(2)}$$

$$z^{(2)} = \tfrac{1}{3}(1 + \zeta t_1 + \zeta^2 t_1^2) \qquad t_1^2 = z^{(0)} + \zeta^2 z^{(1)} + \zeta z^{(2)}$$

<u>For general r</u>

$$z^{(j)} = \frac{1}{r} \sum_{l=0}^{r-1} \zeta^{-lj} t_1^l, \qquad t_1^j = \sum_{l=0}^{r-1} \zeta^{lj} z^{(l)}.$$

Conversion 2: $\mathbb{H}^0$ to $\mathcal{D}^0$

Let $\kappa, c_1, c_2, \ldots, c_{r-1} \in \mathbb{C}$.

$\mathbb{H}^0$ is the algebra generated by x_1^{-1}, x_1, y_1, t_1 with relations

$$x_1^{-1}x_1 = x_1x_1^{-1} = 1, \quad t_1x_1 = \zeta x_1 t_1, \quad t_1 y_1 = \zeta y_1 t_1$$

$$y_1x_1 = x_1y_1 + \kappa - \sum_{l=1}^{r-1} c_l(1-\zeta^{-l})t_1^l$$

$$= x_1y_1 + \kappa - \sum_{j=0}^{r-1}(k_j - k_{j-1})z^{(j)} \qquad \text{[Gri (4,12)]}$$

where $k_j = \frac{1}{r}\sum_{l=0}^{r-1}\zeta^{lj}c_l$.

Let $V = \mathbb{C}^1 = \mathbb{C}$

Let $V^0 = V^{reg} = \mathbb{C}^\times = \mathbb{D}^\times = \mathbb{C} - \{0\}$.

Gri Yaping Ting Gufang

$\mathcal{D}^0 = \mathcal{D}(V^0) \rtimes W$ is the algebra generated by $x_1^{-1}, x_1, \partial_1, t_1$ with relations

$$x_1^{-1}x_1 = x_1x_1^{-1} = 1, \quad t_1x_1 = \zeta x_1 t_1, \quad t_1y_1 = \zeta y_1 t_1$$

$$\partial_1 x_1 = x_1\partial_1 + 1$$

In fact $\mathbb{H}^0$ and $\mathcal{D}^0$ are <u>the</u> <u>same</u>.

$$y_1 = \partial_1 + \sum_{i=0}^{r-1} r(k_i - k_0)\frac{1}{x_1}z^{(i)}, \quad \partial_1 = y_1 - \sum_{i=0}^{r-1}\frac{1}{x_1}rk_iz^{(i)}.$$

The $\mathcal{D}^0$-module $\Delta(E)^0$

Let $\Delta(E)^0$ be the $\mathbb{H}^0$-module generated by e_0, e_1, e_2 with relations

$$t_1 e_j = \zeta^j e_j \quad \text{and} \quad y_1 e_j = 0.$$

Then

$$\Delta(E)^0 \text{ has basis } \left\{ x_1^a e_j \;\middle|\; \begin{array}{l} a \in \mathbb{Z} \\ j \in \{0,1,\dots,r-1\} \end{array} \right\}$$

An element of $\Delta(E)^0$ looks like

$$p = p_0 e_0 + p_1 e_1 + p_2 e_2 = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}$$

where $p_0, p_1, p_2 \in \mathbb{C}[x_1, x_1^{-1}]$

Then

$$\partial_1 p = \partial_1 \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} - \frac{k_0}{x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_1} - \frac{k_1}{x_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_1} - \frac{k_2}{x_1} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}$$

Horizontal sections

The vector space of horizontal sections of $\Delta(E)^\circ$ is

$$KZ(\Delta(E)^\circ) = \{ p \in \Delta(E)^\circ \mid \partial_1 p = 0 \}$$

So

$$p = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} \in KZ(\Delta(E)^\circ) \quad \text{if}$$

$$\frac{\partial p_0}{\partial x_1} = \frac{r k_0}{x_1} p_0, \qquad \frac{\partial p_1}{\partial x_1} = \frac{r k_1}{x_1} p_1, \qquad \frac{\partial p_2}{\partial x_1} = \frac{r k_2}{x_1} p_2$$

So

$$\begin{aligned} p &= p_0 e_0 + p_1 e_1 + p_2 e_2 \\ &= C_0 x_1^{r k_0} e_0 + C_1 x_1^{r k_1} e_1 + C_2 x_1^{r k_2} e_2 \end{aligned}$$

where $C_0, C_1, C_2 \in \mathbb{C}$.

So

$$KZ(\Delta(E)^\circ) = \operatorname{span}\{ x_1^{r k_0} e_0, x_1^{r k_1} e_1, x_1^{r k_2} e_2 \}.$$

Allowing denominators

$V = \mathbb{C}$ and $V^\circ = \mathbb{C} - \{0\}$

$\mathbb{C}[V] = \mathbb{C}[x_1]$ and $\mathbb{C}[V^\circ] = \mathbb{C}[x_1, x_1^{-1}]$.

Monodromy in V^0

The fundamental groupoid of V^0 is

$$P(V^0; a,b) = \left\{ \gamma : \mathbb{R}_{[0,1]} \to V^0 \;\middle|\; \begin{matrix} \gamma(0)=a \\ \gamma(1)=b \end{matrix} \right\}$$

Let $\gamma \in P(V^0; a,b)$.

Let $f_0, f_1, f_2 \in KZ(\Delta(E)^0)$ be such that

$$f_0(\gamma(0)) = f_0(a) = e_0,$$
$$f_1(\gamma(0)) = f_1(a) = e_1,$$
$$f_2(\gamma(0)) = f_2(a) = e_2.$$

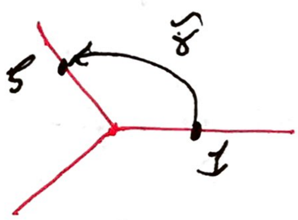
(initial conditions for differential equations)

The monodromy of γ is the matrix $S(\gamma) \in End(E)$ given by

$$S(\gamma)e_0 = f_0(\gamma(1)) = f_0(b),$$
$$S(\gamma)e_1 = f_1(\gamma(1)) = f_1(b),$$
$$S(\gamma)e_2 = f_2(\gamma(1)) = f_2(b).$$

Example $\gamma \in P(V^0; 1, \zeta)$

$$\gamma : \mathbb{R}_{[0,1]} \to V^0$$
$$t \mapsto e^{2\pi i t/r}$$



has $\gamma(0)=1$ and $\gamma(1) = e^{2\pi i/r}$.

10.12.2024
KZ
5.5

then

$$f_0 = x_1^{rk_0} e_0 , \quad f_1 = x_1^{rk_1} e_1 , \quad f_2 = x_1^{rk_2} e_2$$

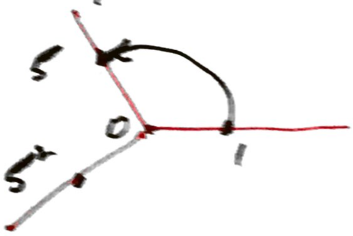
since $f_0(\gamma(0)) = f_0(1) = 1^{rk_0} e_0 = e_0$, and similarly for f_1 and f_2. Then

$$S(\gamma) = \begin{pmatrix} e^{2\pi i k_0} & 0 & 0 \\ 0 & e^{2\pi i k_1} & 0 \\ 0 & 0 & e^{2\pi i k_2} \end{pmatrix}$$

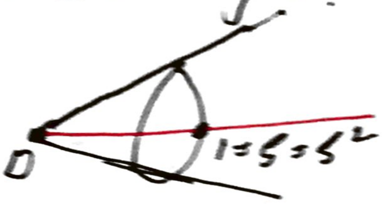
since $f_0(\gamma(1)) = f_0(e^{2\pi i/r}) = (e^{2\pi i/r})^{rk_0} e_0 = e^{2\pi i k_0} e_0$ and similarly for f_1 and f_2.

Monodromy on V°/W

t_1 acts on V° by multiplication by ζ.



and $V^\circ/W =$



t_1 acts on $KZ(\Delta(E)^\circ) = \mathrm{span}\{e_0, e_1, e_2\}$ by

$$t_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$$

The <u>fundamental group</u> is

$$\pi_1(V^\circ/W, 1) = \left\{ \gamma : \mathbb{R}_{[0,1]} \to V^\circ/W \;\middle|\; \begin{matrix} \gamma(0)=1 \\ \gamma(1)=1 \end{matrix} \right\}$$

Let $\gamma \in \pi_1(V^\circ/W, 1)$. The <u>monodromy ~~of~~ matrix</u> is the matrix

$$T(\gamma) = S(\tilde{\gamma})\,(\bar{\gamma})^{-1}$$

where

$\tilde{\gamma} \in P(V^\circ; 1, \tilde{\gamma}(1))$ and $\bar{\gamma} \in W$ such that $\tilde{\gamma}/W = \gamma$.

<u>For example</u> If $\gamma : \mathbb{R}_{[0,1]} \to V^\circ/W$, $t \mapsto e^{2\pi i t/r}$ then

$$T(\gamma) = \begin{pmatrix} e^{2\pi i k_0} & 0 & 0 \\ 0 & e^{2\pi i k_1} & 0 \\ 0 & 0 & e^{2\pi i k_2} \end{pmatrix} \begin{pmatrix} 1 & & \\ & \zeta^{-1} & \\ & & \zeta^{-2} \end{pmatrix} = \begin{pmatrix} e^{2\pi i k_0} & 0 & 0 \\ 0 & \zeta^{-1} e^{2\pi i k_1} & 0 \\ 0 & 0 & \zeta^{-2} e^{2\pi i k_2} \end{pmatrix}$$