13.12.2021 ①
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The reflection group $G(3,3,2)$

Let $\zeta = e^{2\pi i/3}$.

The reflection representation $\mathfrak{a}^*$

$\mathfrak{a}^* = \mathrm{span}\{\varepsilon_1, \varepsilon_2\}$ is different from $\mathfrak{a} = \mathrm{span}\{\varepsilon_1^\vee, \varepsilon_2^\vee\}$.

They are connected by $\langle\, , \rangle : \mathfrak{a}^* \otimes \mathfrak{a} \to \mathbb{C}$

given by $\langle \varepsilon_i, \varepsilon_j^\vee \rangle = \delta_{ij}$.

The reflection group $G(3,3,2)$ is $\mathfrak{a}^*$ with the action of $W_{fin} = \{1, s_0s_1, s_1s_0, s_0, s_1, s_2\}$

where, in the basis $\{\varepsilon_1, \varepsilon_2\}$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad s_0s_1 = \begin{pmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{pmatrix} \qquad s_1s_0 = \begin{pmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{pmatrix}$$

$$s_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad s_1 = \begin{pmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{pmatrix}$$

Note: (a) $\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} s_0 \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \zeta \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \zeta^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix} = s_1$

(b) s_0, s_1, s_2 are conjugate in W_{fin}!

$$s_1 s_0 s_1^{-1} = \begin{pmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{pmatrix} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}\begin{pmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{pmatrix} = s_2 = \begin{pmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{pmatrix}$$

$$s_2 s_0 s_2^{-1} = \begin{pmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{pmatrix} = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta \end{pmatrix}\begin{pmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{pmatrix} = s_1 = \begin{pmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{pmatrix}.$$

Reflections in $G(3,3,2)$

The <u>set of reflections</u> is

$$R = \{s_0, s_1, s_2\}.$$

$s_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ fixes $\mathfrak{h}^{\alpha_0^\vee} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = \mathrm{span}\{\varepsilon_1 + \varepsilon_2\}$

$s_1 = \begin{pmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{pmatrix}$ fixes $\mathfrak{h}^{\alpha_1^\vee} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ \zeta^2 \end{pmatrix}\right\} = \mathrm{span}\{\varepsilon_1 + \zeta^2 \varepsilon_2\}$

$s_2 = \begin{pmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{pmatrix}$ fixes $\mathfrak{h}^{\alpha_2^\vee} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ \zeta \end{pmatrix}\right\} = \mathrm{span}\{\varepsilon_1 + \zeta\varepsilon_2\}$

Then

$$\mathfrak{h}^{\alpha_0^\vee} = \{\mu \in \mathfrak{a}^* \mid \langle \mu, \alpha_0^\vee \rangle = 0\}, \text{ where } \alpha_0^\vee = \varepsilon_1^\vee - \varepsilon_2^\vee$$

$$\mathfrak{h}^{\alpha_1^\vee} = \{\mu \in \mathfrak{a}^* \mid \langle \mu, \alpha_1^\vee \rangle = 0\}, \text{ where } \alpha_1^\vee = \varepsilon_1^\vee - \zeta\varepsilon_2^\vee$$

$$\mathfrak{h}^{\alpha_2^\vee} = \{\mu \in \mathfrak{a}^* \mid \langle \mu, \alpha_2^\vee \rangle = 0\}, \text{ where } \alpha_2^\vee = \varepsilon_1^\vee - \zeta^2\varepsilon_2^\vee.$$

For $\mu = \mu_1\varepsilon_1 + \mu_2\varepsilon_2 \in \mathfrak{a}^*$,

$$s_0\mu = \mu - \langle \mu, \alpha_0^\vee \rangle \alpha_0, \text{ where } \alpha_0 = \varepsilon_1 - \varepsilon_2$$

$$s_1\mu = \mu - \langle \mu, \alpha_1^\vee \rangle \alpha_1, \text{ where } \alpha_1 = \varepsilon_1 - \zeta^2\varepsilon_2$$

$$s_2\mu = \mu - \langle \mu, \alpha_2^\vee \rangle \alpha_2, \text{ where } \alpha_2 = \varepsilon_1 - \zeta\varepsilon_2$$

The <u>set of reflecting hyperplanes</u> is

$$\mathcal{A} = \{\mathfrak{h}^{\alpha_0^\vee}, \mathfrak{h}^{\alpha_1^\vee}, \mathfrak{h}^{\alpha_2^\vee}\}.$$

The Coxeter group G

Let G be the abstract group generated by symbols t_{s_0}, t_{s_1} with relations

$$t_{s_0}^2 = 1, \quad t_{s_1}^2 = 1, \quad t_{s_0} t_{s_1} t_{s_0} = t_{s_1} t_{s_0} t_{s_1}$$

G is <u>not</u> <u>a</u> <u>reflection group</u>. A <u>reflection group</u> is a <u>pair</u> $(\mathfrak{h}^*, W_{fin})$ where $\mathfrak{h}^*$ is a vector space and W_{fin} is a group of linear transformations of $\mathfrak{h}^*$ generated by reflections.

$$W_{fin} \cong G \quad \text{as groups}$$

but the statement $W_{fin} \cong G$ as reflection groups makes no sense.

<u>The</u> <u>dual reflection representation $\mathfrak{h}$</u>

The action of W_{fin} on $\mathfrak{h}$ is defined by

$$\langle w\mu, \lambda^\vee \rangle = \langle \mu, w^{-1}\lambda^\vee \rangle$$

for $w \in W_{fin}$, $\mu \in \mathfrak{h}^*$, $\lambda^\vee \in \mathfrak{h}$. In the basis $\{\varepsilon_1^\vee, \varepsilon_2^\vee\}$ of $\mathfrak{h}$ the group W_{fin} acts by

$$1^\vee = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad s_0^\vee s_1^\vee = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta \end{pmatrix} \qquad s_1^\vee s_0^\vee = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}$$

$$s_0^\vee = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad s_1^\vee = \begin{pmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{pmatrix} \qquad s_2^\vee = \begin{pmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{pmatrix}$$

The rational Cherednik algebra $\tilde{H}$

The xs and ys Let $\kappa, c_0 \in \mathbb{C}$.
Define

$$x_1 = x_{\varepsilon_1}, \quad x_2 = x_{\varepsilon_2}, \quad \mu_1 x_1 + \mu_2 x_2 = x_{\mu_1\varepsilon_1 + \mu_2\varepsilon_2}$$

$$y_1 = y_{\varepsilon_1^\vee}, \quad y_2 = y_{\varepsilon_2^\vee} \quad \lambda_1 y_1 + \lambda_2 y_2 = y_{\lambda_1\varepsilon_1^\vee + \lambda_2\varepsilon_2^\vee}$$

for $\mu_1, \mu_2 \in \mathbb{C}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$.

The rational Cherednik algebra $\tilde{H}$ is generated by

$$x_\mu, \ y_{\lambda^\vee}, \ t_w \qquad \text{for } \mu \in \mathfrak{a}^*, \ \lambda^\vee \in \mathfrak{a}, \ w \in W_{\mathrm{fin}}$$

with relations

$$t_w t_v = t_{wv}, \quad t_w x_\mu = x_{w\mu} t_w, \quad t_w y_{\lambda^\vee} = y_{w\lambda^\vee} t_w$$

$$x_1 x_2 = x_2 x_1, \qquad y_1 y_2 = y_2 y_1$$

$$y_1 x_1 = x_1 y_1 + \kappa - c_0 (t_{s_0} + t_{s_1} + t_{s_2})$$

$$y_2 x_2 = x_2 y_2 + \kappa - c_0 (t_{s_0} + t_{s_1} + t_{s_2})$$

$$y_1 x_2 = x_2 y_1 + c_0 (t_{s_0} + \zeta^2 t_{s_1} + \zeta t_{s_2})$$

$$y_2 x_1 = x_1 y_2 + c_0 (t_{s_0} + \zeta t_{s_1} + \zeta^2 t_{s_2}).$$

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<u>To get $\tilde{H}^0$</u> we allow also

$$\frac{1}{x_{\alpha_0}x_{\alpha_1}x_{\alpha_2}} = \frac{1}{x_{\xi_1-\xi_2}\,x_{\xi_1-\zeta\xi_2}\,x_{\xi_1-\zeta^2\xi_2}}.$$

In other words we add

$$\frac{1}{(x_1-x_2)(x_1-\zeta x_2)(x_1-\zeta^2 x_2)}$$

$$\frac{x_1-x_2}{(x_1-x_2)(x_1-\zeta x_2)(x_1-\zeta^2 x_2)} = \frac{1}{(x_1-\zeta x_2)(x_1-\zeta^2 x_2)}$$

$$\frac{x_1-\zeta x_2}{(x_1-x_2)(x_1-\zeta x_2)(x_1-\zeta^2 x_2)} = \frac{1}{(x_1-x_2)(x_1-\zeta x_2)}$$

and also

$$\frac{1}{(x_1-x_2)(x_1-\zeta^2 x_2)},\quad \frac{1}{x_1-x_2},\quad \frac{1}{x_1-\zeta x_2},\quad \frac{1}{x_1-\zeta^2 x_2}$$

(an no other denominators). These are what we get by adding

$$\frac{1}{x_{\alpha_0}x_{\alpha_1}x_{\alpha_2}} = \frac{1}{(x_1-x_2)(x_1-\zeta x_2)(x_1-\zeta^2 x_2)}.$$

The fundamental groupoid

$$\mathfrak{h}^* = \mathrm{span}\{\varepsilon_1, \varepsilon_2\} = \left\{ \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} \,\middle|\, \mu_1, \mu_2 \in \mathbb{C} \right\} = \mathbb{C}^2$$

$$\mathfrak{h}^\circ = \mathfrak{h}^* - \left(\mathfrak{h}^{\alpha_0^\vee} \cup \mathfrak{h}^{\alpha_1^\vee} \cup \mathfrak{h}^{\alpha_2^\vee}\right)$$

$$= \left\{ \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} \,\middle|\, \begin{matrix}\mu_1 \neq \mu_2,\ \mu_1 \neq \zeta\mu_2,\ \mu_1 \neq \zeta^2\mu_2 \\ \mu_1, \mu_2 \in \mathbb{C}\end{matrix} \right\}$$

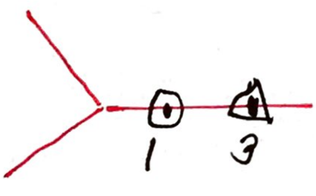
$\mathfrak{h}^*$ has basepoint $a_0 = \begin{pmatrix}1\\2\end{pmatrix} = \varepsilon_1 + 2\varepsilon_2$

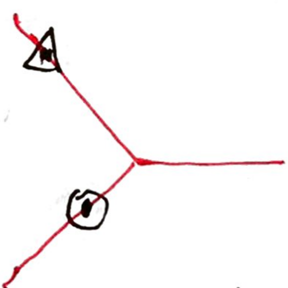
How to draw points and paths in $\mathfrak{h}^\circ$?

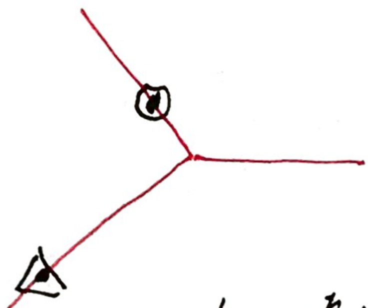
A point in $\mathbb{C}^2$ is a pair of points in $\mathbb{C}$

i.e. two bugs on a plane

The basepoint $a_0 = \begin{pmatrix}1\\3\end{pmatrix}$







$$s_0 a_0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

$$s_1 a_0 = \begin{pmatrix}0 & \zeta\\ \zeta^2 & 0\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}3\zeta\\ \zeta^2\end{pmatrix}$$

$$s_2 a_0 = \begin{pmatrix}0 & \zeta^2\\ \zeta & 0\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}3\zeta^2\\ \zeta\end{pmatrix}$$

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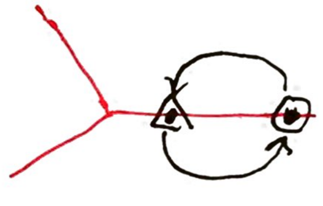
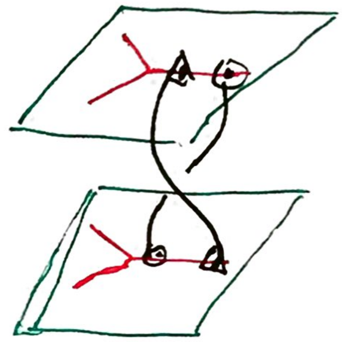
Paths in $\mathfrak{a}^{\circ}$

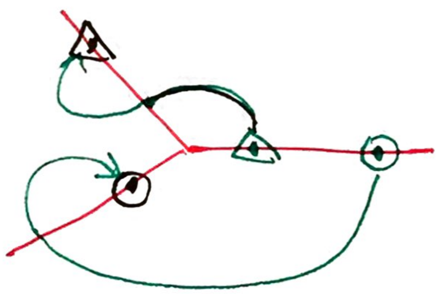
Path from a_0 to $s_0 a_0$





$$\gamma_0 : \mathbb{R}_{[0,1]} \longrightarrow \mathfrak{a}^{\circ}$$
$$t \longmapsto \begin{pmatrix} 2+e^{i\pi + i\pi t} \\ 2+e^{i\pi t} \end{pmatrix}$$

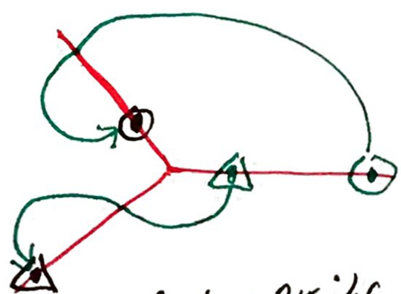
Path from a_0 to $s_1 a_0$



$$\gamma_1 : \mathbb{R}_{[0,1]} \longrightarrow \mathfrak{a}^{\circ}$$

given by
$$\gamma_1(t) = \begin{cases} \begin{pmatrix} e^{2\pi i t/3} \\ 3e^{-2\pi i t/3} \end{pmatrix}, & \text{if } t \in \mathbb{R}_{[0,\frac{1}{2}]}, \\ \begin{pmatrix} \zeta\left(2+e^{2i\pi(1-t)}\right) \\ \zeta^2\left(2+e^{2i\pi(\frac{1}{2}-t)}\right) \end{pmatrix}, & \text{if } t \in \mathbb{R}_{[\frac{1}{2},1]}. \end{cases}$$

Path from a_0 to $s_2 a_0$



$$\gamma_2 : \mathbb{R}_{[0,1]} \longrightarrow \mathfrak{a}^{\circ}$$

given by
$$\gamma_2(t) = \begin{cases} \begin{pmatrix} e^{-2\pi i t/3} \\ 3e^{2\pi i t/3} \end{pmatrix}, & \text{if } t \in \mathbb{R}_{[0,\frac{1}{2}]}, \\ \begin{pmatrix} \zeta^2\left(2+e^{2i\pi t}\right) \\ \zeta\left(2+e^{2i\pi(-\frac{1}{2}+t)}\right) \end{pmatrix}, & \text{if } t \in \mathbb{R}_{[\frac{1}{2},1]} \end{cases}$$