W_{fin}-modules to H_{fin}-modules

$$\{W_{fin}\text{-modules}\} \xrightarrow{\Delta} \{\mathbb{H}^0\text{-modules}\} \xrightarrow{KZ} \{H_{fin}\text{-modules}\}$$

$$E \longmapsto \Delta(E) \longmapsto E_q$$

<u>Example Type $G(r,1,1)$</u>: Let $\zeta = e^{2\pi i/r}$.

As a group W_{fin} is generated by t_1 with relation

$$t_1^r = 1.$$

The <u>Hecke algebra</u> H_{fin} is generated by T_1 with

$$(T_1 - q_0)(T_1 - q_1)\cdots(T_1 - q_{r-1}) = 0.$$

For $j \in \{0, 1, \ldots, r-1\}$ let

$$E^{(j)} = \text{span}\{e_1^{(j)}\} \quad \text{with} \quad t_1 e_1^{(j)} = \zeta^j e_1^{(j)}$$

For $j \in \{0, 1, \ldots, r-1\}$ let

$$E^{(j)} = \text{span}\{e_1^{(j)}\} \quad \text{with} \quad T_1 e_1^{(j)} = q_j e_1^{(j)}$$

<u>Theorem</u>

(a) $\{E^{(j)} \mid j \in \{0, 1, \ldots, r-1\}\}$ are the simple W_{fin}-modules.

(b) $\{E_q^{(j)} \mid j \in \{0, 1, \ldots, r-1\}\}$ are the simple H_{fin}-modules.

Example: Type $G(r,r,2)$. Let $\zeta = e^{2\pi i/r}$

$$G(r,r,2) = \left\{ \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix}, \begin{pmatrix} 0 & \zeta^k \\ \zeta^{-k} & 0 \end{pmatrix} \;\middle|\; k \in \{0,1,\ldots,r-1\} \right\}$$

As a group W_{fin} is generated by s_1 and s_2 with

$$s_1^2 = 1, \quad s_2^2 = 1 \quad \text{and} \quad \underbrace{s_1 s_2 s_1 \cdots}_{r \text{ factors}} = \underbrace{s_2 s_1 s_2 \cdots}_{r\text{-factors}}$$

Let $p, q \in \mathbb{C}^\times$. The <u>Hecke algebra</u> $H_{r,r,2}$ is generated by T_1 and T_2 with

$$\underbrace{T_1 T_2 T_1 \cdots}_{r \text{ factors}} = \underbrace{T_2 T_1 T_2 \cdots}_{r\text{-factors}} \quad \text{and} \quad \begin{aligned} (T_1 - p)(T_1 + p^{-1}) &= 0 \\ (T_1 - q)(T_1 + q^{-1}) &= 0. \end{aligned}$$

Note: If r is odd then $T_2 = \underbrace{(T_1 T_2 \cdots)}_{r-1 \text{ factors}} T_1 \underbrace{(T_1 T_2 \cdots)}_{r-1 \text{ factors}}^{-1}$ and so $p = q$.

<u>Theorem</u> The irreducible representations of W_{fin} are given as follows:

$E^{++} = \mathrm{span}\{e^{++}\}$ with $s_1 e^{++} = e^{++}$, $s_2 e^{++} = e^{++}$

$E^{--} = \mathrm{span}\{e^{--}\}$ with $s_1 e^{--} = -e^{--}$, $s_2 e^{--} = -e^{--}$

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and, when r is even,

$$E^{+-} = \mathrm{span}\{e^{+-}\} \text{ with } s_1 e^{+-} = e^{+-}, \quad s_2 e^{+-} = -e^{+-}$$

$$E^{-+} = \mathrm{span}\{e^{-+}\} \text{ with } s_1 e^{-+} = -e^{-+}, \quad s_2 e^{-+} = e^{-+}$$

and $E^{(j)}$ for $j \in \mathbb{Z}_{[0, r/2)}$ given by

$$E^{(j)} = \mathrm{span}\{e_1^{(j)}, e_2^{(j)}\} \text{ with}$$

$$E^{(j)}(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } E^{(j)}(s_2) = \begin{pmatrix} 0 & \zeta^{j} \\ \zeta^{-j} & 0 \end{pmatrix}$$

in the basis $\{e_1^{(j)}, e_2^{(j)}\}$.

__Theorem__ The irreducible representations of H_{im} are given as follows:

$E_q^{(j)}$ for $j \in \mathbb{Z}_{[0, r/2)}$ given by

$$E_q^{(j)} = \mathrm{span}\{e_1^{(j)}, e_2^{(j)}\} \text{ with}$$

$$E_q^{(j)}(T_1) = \begin{pmatrix} a & (1+ad) \\ 1 & d \end{pmatrix} \text{ and } E_q^{(j)} = \begin{pmatrix} -d\zeta^{j} & (1+ad)\zeta^{j} \\ \zeta^{-j} & -a\zeta^{-j} \end{pmatrix}$$

where

$$a = \frac{(p-p^{-1})\zeta + (q-q^{-1})}{\zeta - \zeta^{-1}} \text{ and } d = \frac{(p-p^{-1})\zeta^{-1} + (q-q^{-1})}{\zeta^{-1} - \zeta}$$

and

$E_1^{++} = \mathrm{span}\{e^{++}\}$ with $T_1 e^{++} = p e^{++}$, $T_2 e^{++} = q e^{++}$

$E_1^{--} = \mathrm{span}\{e^{--}\}$ with $T_1 e^{--} = -p^{-1} e^{--}$, $T_2 e^{--} = -q^{-1} e^{--}$

and, if r is even

$E_2^{+-} = \mathrm{span}\{e^{+-}\}$ with $T_1 e^{+-} = p e^{+-}$, $T_2 e^{+-} = -q^{-1} e^{+-}$

$E_2^{-+} = \mathrm{span}\{e^{-+}\}$ with $T_1 e^{-+} = -p^{-1} e^{-+}$, $T_2 e^{-+} = q e^{-+}$

<u>Note:</u> $E^{(j)}(s_1 s_2) = \begin{pmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{pmatrix}$ and $E_2^{(j)}(T_1 T_2) = \begin{pmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{pmatrix}$

and $(T_1 T_2)^r$ commutes with all elements of $H_{r,r,2}$.

<u>Example Case $G(1,1,n)$</u>

$G(1,1,n) = \{n \times n \text{ permutation matrices}\}$

<u>Theorem</u> The symmetric group S_n has a presentation with generators $s_1, \ldots, s_{n-1}$ and relations

$$s_k s_l = s_l s_k \qquad \text{and} \qquad s_i^2 = 1$$
$$s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$$

for $k, l \in \{1, \ldots, n-1\}$ with $l \notin \{k+1, k-1\}$
$j \in \{1, \ldots, n-2\}$ and $i \in \{1, \ldots, n-1\}$

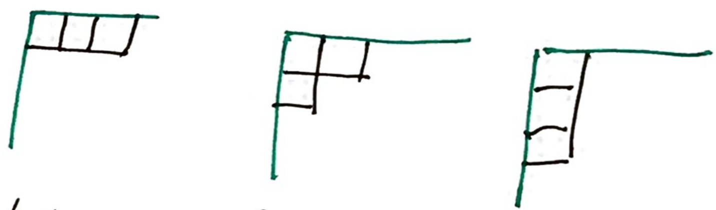
Let $q \in \mathbb{C}^\times$.
The <u>Hecke algebra</u> $H_{1,1,n}$ is generated by $T_1, T_2, \ldots, T_{n-1}$ with relations

$$T_kT_l = T_lT_k$$
$$T_jT_{j+1}T_j = T_{j+1}T_jT_{j+1} \quad \text{and} \quad (T_i - q)(T_i + q^{-1}) = 0$$

for $k, l \in \{1, \ldots, n-1\}$ with $l \notin \{k+1, k-1\}$, $j \in \{1, \ldots, n-2\}$ and $i \in \{1, \ldots, n-1\}$.

A <u>partition of n</u> is a collection of n boxes in a corner (gravity goes up and left).

<u>Example: $n=3$</u>



A <u>standard tableau of shape λ</u> is a filling T of the boxes of λ with $\{1, 2, \ldots, n\}$ such that
(a) the rows of T are increasing left to right
(b) the columns of T are increasing top to bottom

1	2	5	6	10
3	4	7	12	15
8	9	14		
11	16	18		
13				
17				

A Standard tableau of shape $\lambda = (553311)$.

0	1	2	3
-1	0	1	2
-2	-1	0	
-3	-2	-1	
-4			
-5			

Contents of boxes

Let

$T(i)$ be the box containing i in T

$c(b) = j - i$ if b is in position (i,j) in λ.

<u>Theorem</u>

(a) The irreducible representations E^λ of S_n are indexed by partitions λ with n boxes.

(b) $\dim(E^\lambda) = \#\{\text{standard tableaux of shape } \lambda\}$

(c) $E^\lambda = \mathrm{span}\left\{ e^\lambda_T \;\middle|\; \begin{matrix} T \text{ is a standard tableau} \\ \text{of shape } \lambda \end{matrix} \right\}$

with S_n-action given by

$$s_i e^\lambda_T = \left(\frac{1}{c(T(i+1)) - c(T(i))}\right) v_T + \left(1 + \frac{1}{c(T(i+1)) - c(T(i))}\right) e^\lambda_{s_iT}$$

where

$c(T(i))$ = content of box containing i in T

s_iT is the same as T except i and $i+1$ are switched

$e^\lambda_{s_iT} = 0$ if s_iT is not standard.

<u>Example: $n=3$</u>

$$E^{\square\square\square}(s_1) = (1) \qquad E^{(2,1)}(s_1) = \begin{array}{cc} \begin{smallmatrix}1&2\\3&\end{smallmatrix} & \begin{smallmatrix}1&3\\2&\end{smallmatrix} \\ \multicolumn{2}{c}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \end{array} \qquad E^{(1,1,1)}(s_1) = (-1)$$

$$E^{\square\square\square}(s_2) = (1) \qquad E^{(2,1)}(s_2) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \qquad E^{(1,1,1)}(s_2) = (-1).$$

<u>Theorem</u> Assume $[n]! \neq 0$

(a) The irreducible representations E_q^λ of $H_{1,1,n}$ are indexed by partitions λ with n boxes.

(b) $\dim(E_q^\lambda) = \#\{\text{standard tableaux } T \text{ of shape } \lambda\}$.

(c) $E_q^\lambda = \text{span}\left\{ e_T^\lambda \ \middle|\ \begin{array}{l} T \text{ is a standard tableau} \\ \text{of shape } \lambda \end{array} \right\}$

with $H_{1,1,n}$-action given by

$$T_i e_T^\lambda = \left(\frac{q-q^{-1}}{1-q^{2(c(T(i+1))-c(T(i)))}}\right) e_T^\lambda + \left(q^{-1} + \frac{q-q^{-1}}{1-q^{2(c(T(i))-c(T(i+1)))}}\right) e_{s_iT}^\lambda$$

<u>Example:</u> $n=3$

$E_q^{(3)}(T_1) = (q)$

$E_q^{(3)}(T_2) = (q)$

$$E_q^{(2,1)}(T_2) = \begin{pmatrix} \frac{q-q^{-1}}{1-q^4} & q^{-1} + \frac{q-q^{-1}}{1-q^{-4}} \\ q^{-1} + \frac{q-q^{-1}}{1-q^4} & \frac{q-q^{-1}}{1-q^{-4}} \end{pmatrix}$$

$$E_q^{(2,1)}(T_1) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$E_q^{(1,1,1)}(T_1) = (-q^{-1})$

$E_q^{(1,1,1)}(T_2) = (-q^{-1})$.