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Rep. Thy Seminar ①
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# The DAWG $\widetilde{W}$ of type $GL_n$

The symmetric group $S_n$ acts on

$$\mathbb{Z}^n = \mathbb{Z}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n\} \quad \text{where} \quad \varepsilon_i = (0, \ldots 0, \overset{i\text{th}}{1}, 0, \ldots 0)$$

by permuting $\varepsilon_1, \ldots, \varepsilon_n$.

The <u>DAWG</u> $\widetilde{W}$ is generated by $q^{k}$ and

$x_\mu$ for $\mu \in \mathbb{Z}^n$, $y_\lambda$ for $\lambda \in \mathbb{Z}^n$, $w$ for $w \in S_n$

with

$$q \in Z(\widetilde{W}), \qquad S_n = \{w \mid w \in S_n\} \text{ a subgroup},$$

$$x_\mu x_\nu = x_{\mu+\nu}, \quad y_\lambda y_\gamma = y_{\lambda+\gamma}, \quad x_\mu y_\lambda = q^{\langle \mu, \lambda \rangle} y_\lambda x_\mu$$

$$w x_\lambda = x_{w\lambda} w \quad \text{and} \quad w y_\lambda = y_{w\lambda} w.$$

Define

$$t_\lambda = q^{-\frac{1}{2}|\lambda|^2} x_\lambda y_\lambda, \quad \text{for } \lambda \in \mathbb{Z}^n.$$

Then

$$t_\lambda t_\mu = t_{\lambda+\mu} \quad \text{and} \quad w t_\lambda = t_{w\lambda} w.$$

Let

$$W_X = \{x_\lambda w \mid \lambda \in \mathbb{Z}^n, w \in S_n\},$$

$$W = \{t_\lambda w \mid \lambda \in \mathbb{Z}^n, w \in S_n\}, \qquad D = \left\{ q^k x_\lambda y_\mu \;\middle|\; \begin{matrix} \lambda, \mu \in \mathbb{Z}^n \\ k \in \mathbb{Z} \end{matrix} \right\}$$

$$W_Y = \{y_\lambda w \mid \lambda \in \mathbb{Z}^n, w \in S_n\},$$

are three affine Weyl groups and a Heisenberg group.

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# Matrix representations of $\widetilde{W}$

## In $GL_{n+2}$

$$x_\mu = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & \mu \\ \hline 0 & 0 & 1 \end{array}\right) \qquad y_\lambda = \left(\begin{array}{c|c|c} 1 & \lambda & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$q^k = \left(\begin{array}{c|c|c} 1 & 0 & k \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \qquad w = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & w & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

## In $GL_{n+4}$

$$x_\mu = \left(\begin{array}{cc|c|cc} 1 & 0 & -\mu & -\frac{1}{2}|\mu|^2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & & 1 & \mu & 0 \\ \hline 0 & & 0 & 1 & 0 \\ & & & 0 & 1 \end{array}\right) \qquad y_\lambda = \left(\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & -\frac{1}{2}|\lambda|^2 \\ \hline 0 & & 1 & 0 & -\lambda \\ \hline 0 & & 0 & 1 & 0 \\ & & & 0 & 1 \end{array}\right)$$

$$q^k = \left(\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & k \\ 0 & 1 & 0 & -k & 0 \\ \hline 0 & & 1 & 0 & 0 \\ \hline 0 & & 0 & 1 & 0 \\ & & & 0 & 1 \end{array}\right) \qquad w = \left(\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & & 1 & 0 & 0 \\ \hline 0 & & 0 & 1 & 0 \\ & & & 0 & 1 \end{array}\right)$$

Let $s_i = (i, i+1) \in S_n$ be the transposition switching $i$ and $i+1$ and define

$$s_\pi = y_{\varepsilon_1} s_1 \cdots s_{n-1}, \qquad s_\pi^\sharp = q^{-\frac{1}{2}} x_{\varepsilon_1} y_{\varepsilon_1} s_1 \cdots s_{n-1}, \qquad s_\pi^\vee = x_{\varepsilon_1} s_1 \cdots s_{n-1}$$

# The DAArt $\hat{\mathcal{B}}$ of type $GL_n$

Use Dynkin diagram notation so that

$\overset{a}{\circ}\!-\!\overset{b}{\circ}$ means $aba = bab$

$\overset{a}{\circ}\ \overset{b}{\circ}$ means $ab = ba$

The <u>DAArt</u> $\hat{\mathcal{B}}$ is given by generators

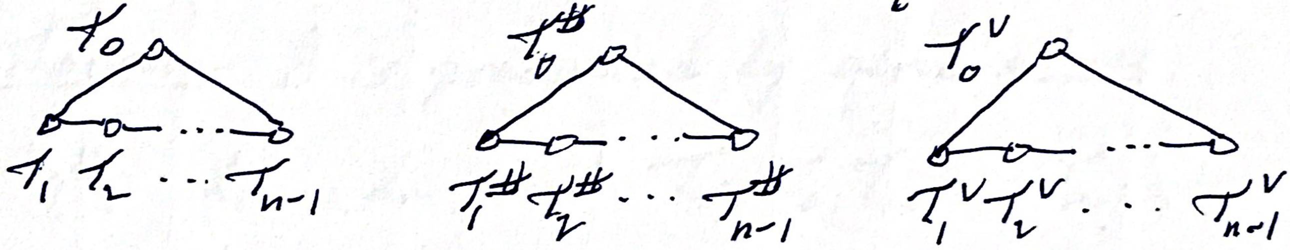
$q^{\frac{1}{2}}$ and $T_\pi, T_0, T_1, \dots T_{n-1},$

$T_\pi^\#, T_0^\#, T_1^\#, \dots, T_{n-1}^\#,$

$T_\pi^\vee, T_0^\vee, T_1^\vee, \dots, T_{n-1}^\vee$

with relations

$q^{1/2} \in Z(\hat{\mathcal{B}})$, $\qquad T_i = T_i^\# = T_i^\vee$ for $i \in \{1, \dots, n-1\}$




$T_\pi T_i T_\pi^{-1} = T_{i+1} \qquad T_\pi^\# T_i^\# (T_\pi^\#)^{-1} = T_{i+1}^\# \qquad T_\pi^\vee T_i^\vee (T_\pi^\vee)^{-1} = T_{i+1}^\vee$

$$T_\pi^{-1} T_\pi^\# (T_\pi^\vee)^{-1} T_1 \cdots T_{n-1} = q^{1/2}$$

$$(T_\pi^\vee)^{-1} T_\pi^\# T_\pi^{-1} T_1^{-1} \cdots T_{n-1}^{-1} = q^{-1/2}$$

$$T_0^\vee T_0^\# T_0 T_1 \cdots T_{n-1} \cdots T_1 = q^{-1}$$

# Relation to DAWG and Cherednik

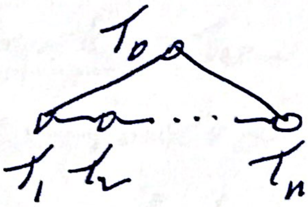
**Theorem** There is a surjective homomorphism

$$\begin{aligned} \hat{\mathcal{B}} &\longrightarrow \hat{W} \\ T_\pi &\longmapsto s_\pi \\ T_\pi^\sharp &\longmapsto s_\pi^\sharp \\ T_\pi^\vee &\longmapsto s_\pi^\vee \\ T_i &\longmapsto s_i \quad \text{for } i \in \{1, \ldots, n-1\}. \end{aligned}$$

with kernel generated by $T_1^2$.

Let
$$X_1 = T_\pi^\vee T_{n-1}^{-1} \cdots T_1^{-1} \quad \text{and} \quad X_j = T_{j-1} X_{j-1} T_{j-1}$$

**Theorem** $\hat{\mathcal{B}}$ is presented by generators
$$q^{\frac{1}{2}}, \quad X_1, \ldots, X_n \quad \text{and} \quad T_\pi, T_0, T_1, \ldots, T_n$$
with relations
$$q^{\frac{1}{2}} \in Z(\hat{\mathcal{B}}), \qquad X_i X_j = X_j X_i$$




and
$$T_\pi X_i T_\pi^{-1} = X_{i+1}$$

$$T_\pi T_i T_\pi^{-1} = T_{i+1} \qquad\qquad T_\pi X_n T_\pi^{-1} = q^{-1} X_1.$$

Also define
$$Y_1 = T_\pi T_{n-1} \cdots T_1 \quad \text{and} \quad Y_j = T_{j-1}^{-1} Y_{j-1}^{+1} T_{j-1}^{-1}$$

## $GL_2(\mathbb{Z})$

$$GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \middle| \begin{array}{l} a,b,c,d \in \mathbb{Z} \\ ad-bc \in \mathbb{Z}^\times \end{array} \right\} \quad \text{with } \mathbb{Z}^\times = \{1,-1\}.$$

Let

$$\sigma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

**Proposition**

(a) $GL_2(\mathbb{Z})$ is presented by generators $\sigma_1, \sigma_2, s$ with relations

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad (\sigma_1\sigma_2\sigma_1)^4 = 1, \quad s^2 = 1,$$

$$s\sigma_1 = \sigma_2^{-1}s \quad \text{and} \quad s\sigma_2 = \sigma_1^{-1}s.$$

(b) $GL_2(\mathbb{Z})$ is presented by generators $\gamma_1, \gamma_2, e$ with relations

$$\gamma_2^3 = \gamma_1^2, \quad \gamma_1^4 = 1, \quad e^2 = 1$$

$$e\gamma_1 = \gamma_1^{-1}e, \quad e\gamma_2 = \gamma_2^{-1}e$$

**Remark** The braid group on 3-strands $B_3$ is presented by generators $g_1, g_2$ with relation $g_1g_2g_1 = g_2g_1g_2$.

**Remark**

$$(g_1g_2g_1)^2 \in Z(B_3) \quad \text{and} \quad \gamma_1^2 = (\sigma_1\sigma_2\sigma_1)^2 \in Z(GL_2(\mathbb{Z})).$$

## Automorphisms

$GL_2(\mathbb{Z})$ embeds into $GL_{n+1}(\mathbb{Z})$ by

$$GL_2(\mathbb{Z}) \hookrightarrow GL_{n+1}(\mathbb{Z})$$

$$u \longmapsto \left(\begin{array}{c|c|c} u & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & (u^{-1})^t \end{array}\right)$$

Conjugation by elements of $GL_2(\mathbb{Z})$ provides automorphisms of $\widetilde{W}$.

<u>Theorem</u> There are automorphisms of $\widetilde{\mathcal{B}}$ given by

$$\sigma_1(q)=q, \quad \sigma_2(q)=q, \quad S(q)=q^{-1}$$

$$\sigma_1(X_1)=X_1, \quad \sigma_2(X_1)=q^{-\frac{1}{2}}X_1Y_1, \quad S(X_1)=T_{w_0}Y_n^{-1}T_{w_0}^{-1}$$

$$\sigma_1(Y_1)=q^{\frac{1}{2}}X_1^{-1}Y_1, \quad \sigma_2(Y_1)=Y_1, \quad S(Y_1)=T_{w_0}X_n^{-1}T_{w_0}^{-1}$$

$$\sigma_1(T_i)=T_i, \quad \sigma_2(T_i)=T_i, \quad S(T_i)=T_i,$$

for $i\in\{1,\dots,n-1\}$. These satisfy

$$\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2, \quad S^2=1,$$

$$S\sigma_1S=\sigma_2^{-1}, \quad S\sigma_2S=\sigma_1^{-1}$$

## Duality automorphisms

The $\vee$-duality automorphism is $\iota: \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ given by

$$\iota(T_\pi) = T_\pi^\vee, \quad \iota(T_\pi^\#) = T_\pi^\#, \quad \iota(T_\pi^\vee) = T_\pi$$

$$\iota(q) = q^{-1} \text{ and } \iota(T_i) = T_i^{-1} \text{ for } i \in \{1, \ldots, n-1\}.$$

Then

$$\left.\begin{array}{l} \iota(X_j) = Y_j \\ \iota(Y_j) = X_j \end{array}\right. \quad \text{for } j \in \{1, \ldots, n\}.$$

The $\leftrightarrow$-duality automorphism is $\eta: \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ given by

$$\eta(T_\pi) = T_\pi^{-1}, \quad \eta(T_\pi^\#) = (T_\pi^\#)^{-1}, \quad \eta(T_\pi^\vee) = (T_\pi^\vee)^{-1}$$

$$\eta(q) = q, \quad \eta(T_i) = T_{n-i} \text{ for } i \in \{1, \ldots, n-1\}$$

Then

$$\left.\begin{array}{l} \eta(X_j) = X_{n-j+1}^{-1}, \\ \eta(Y_j) = Y_{n-j+1}^{-1}, \end{array}\right. \quad \text{for } j \in \{1, \ldots, n\}$$