The modular semigroup $SL_2(\mathbb{Z}_{\geq 0})$

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Theorem

(a) $GL_2(\mathbb{Z})$ is 2 copies of $SL_2(\mathbb{Z})$

(b) $SL_2(\mathbb{Z})$ is 8 copies of $SL_2(\mathbb{Z}_{\geq 0})$

(c) $SL_2(\mathbb{Z}_{\geq 0})$ is a free semigroup on 2 generators

$$GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{Z} \\ ad-bc \in \mathbb{Z}^{\times} \end{matrix} \right\} \quad \text{with } \mathbb{Z}^{\times} = \{1,-1\}.$$

$$\cup|$$

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{Z} \\ ad-bc = 1 \end{matrix} \right\}$$

$$\cup|$$

$$SL_2(\mathbb{Z}_{\geq 0}) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{Z}_{\geq 0} \\ ad-bc = 1 \end{matrix} \right\}$$

Let

$$\urcorner = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \stackrel{=}{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad -1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

(3) $SL_2(\mathbb{Z}_{\geq 0}) = \left\{ a_1 \cdots a_\ell \;\middle|\; \begin{matrix} \ell \in \mathbb{Z}_{\geq 0} \\ a_i \in \{\urcorner, L\} \end{matrix} \right\} = \mathbb{D}$

(2) $SL_2(\mathbb{Z}) = \mathbb{D} \sqcup \mathbb{D}Z \sqcup Z\mathbb{D} \sqcup Z\mathbb{D}Z$
$\sqcup -\mathbb{D} \sqcup -\mathbb{D}Z \sqcup -Z\mathbb{D} \sqcup -Z\mathbb{D}Z$

(1) $GL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \sqcup SL_2(\mathbb{Z}) \stackrel{=}{D}$.

4 different presentations of $GL_2(\mathbb{Z})$

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Generators A $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, with $a, b, c, d \in \mathbb{Z}$
$ad - bc \in \mathbb{Z}^\times$

Relations A

$$\begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + c_1b_2 & a_1c_2 + c_1d_2 \\ b_1a_2 + d_1b_2 & b_1c_2 + d_1d_2 \end{pmatrix}$$

Generators B σ_1, σ_2, e $\quad \sigma_1 = 7,\ \sigma_2 = 6^{-1},\ e = \overline{\text{D}}$

Relations B $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \qquad (\sigma_1\sigma_2\sigma_1)^4 = 1$

$e^2 = 1, \quad e\sigma_1 = \sigma_1^{-1}e, \quad e\sigma_2 = \sigma_2^{-1}e$

Generators C γ_1, γ_2, e $\quad \gamma_1 = Z,\ \gamma_2 = 7L^{-1},\ e = \overline{\text{D}}$

Relations C $\gamma_1^4 = 1, \quad \gamma_2^6 = 1, \quad \gamma_2^3 = \gamma_1^2$

$e^2 = 1, \quad e\gamma_1 = \gamma_1^{-1}e, \quad e\gamma_2 = \gamma_1\gamma_2^{-1}\gamma_1^{-1}e.$

Generators D $x_{12}(k), x_{21}(k)$ with $k \in \mathbb{Z}$, and n and e

Relations D $x_{12}(1)\,x_{21}(-1)\,x_{12}(1) = n$

$x_{12}(k)\,x_{12}(l) = x_{12}(k+l), \qquad x_{21}(k)\,x_{21}(l) = x_{21}(k+l)$

$n^4 = 1, \quad n\,x_{12}(1) = x_{21}(-1)\,n \qquad n\,x_{21}(1) = x_{12}(-1)\,n$

$e^2 = 1, \quad e\,x_{12}(1) = x_{12}(-1)\,n \qquad e\,x_{21}(1) = x_{21}(-1)\,e$

$en = n^{-1}e.$ $\quad x_{12}(1) = 7,\ x_{21}(1) = L$

$n = Z, \quad e = \overline{\text{D}}$

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Normal forms: Writing $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ in Korean

$GL_2(\mathbb{Z})$ is two copies of $SL_2(\mathbb{Z})$

If $ad - bc = -1$ then

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} \bar{O} \quad \text{where} \quad \bar{O} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$SL_2(\mathbb{Z})$ is 8 copies of $SL_2(\mathbb{Z}_{\geq 0})$

If $ad - bc = 1$ then exactly one of

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad -1\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} Z = \begin{pmatrix} -c & a \\ -d & b \end{pmatrix} \qquad -1\begin{pmatrix} a & c \\ b & d \end{pmatrix} Z = \begin{pmatrix} c & -a \\ d & -b \end{pmatrix}$$

$$Z\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} b & d \\ -a & -c \end{pmatrix} \qquad -Z\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -b & -d \\ a & c \end{pmatrix}$$

$$Z\begin{pmatrix} a & c \\ b & d \end{pmatrix} Z = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix} \qquad -1 Z\begin{pmatrix} a & c \\ b & d \end{pmatrix} Z = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

has all entries in $\mathbb{Z}_{\geq 0}$

$SL_2(\mathbb{Z}_{\geq 0})$ is a free semigroup on two generators

$$\begin{pmatrix} 11 & 5 \\ 13 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 5 \\ 13 & 6 \end{pmatrix} L^{-2} = \begin{pmatrix} 1 & 5 \\ 1 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 6 \end{pmatrix} T^{-5} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = L$$

So

$$\begin{pmatrix} 11 & 5 \\ 13 & 6 \end{pmatrix} = L T^5 L^2 = L T T T T T L L$$

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$\Gamma\backslash G/K$: $G = SL_2(\mathbb{R})$, $K = SO_2(\mathbb{R})$
$\Gamma = SL_2(\mathbb{Z})$.

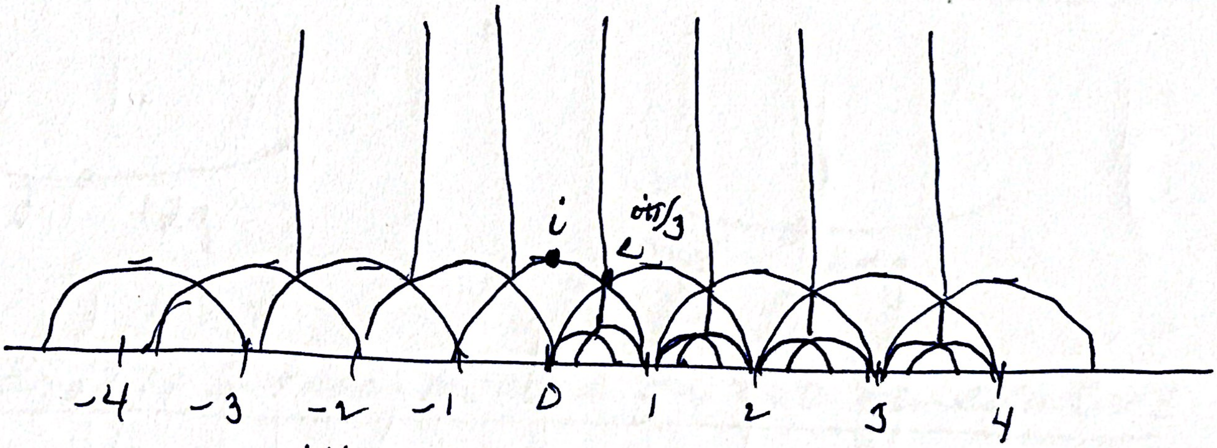
$G/K \xrightarrow{\sim} \mathfrak{H}_1$ where $\mathfrak{H}_1 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

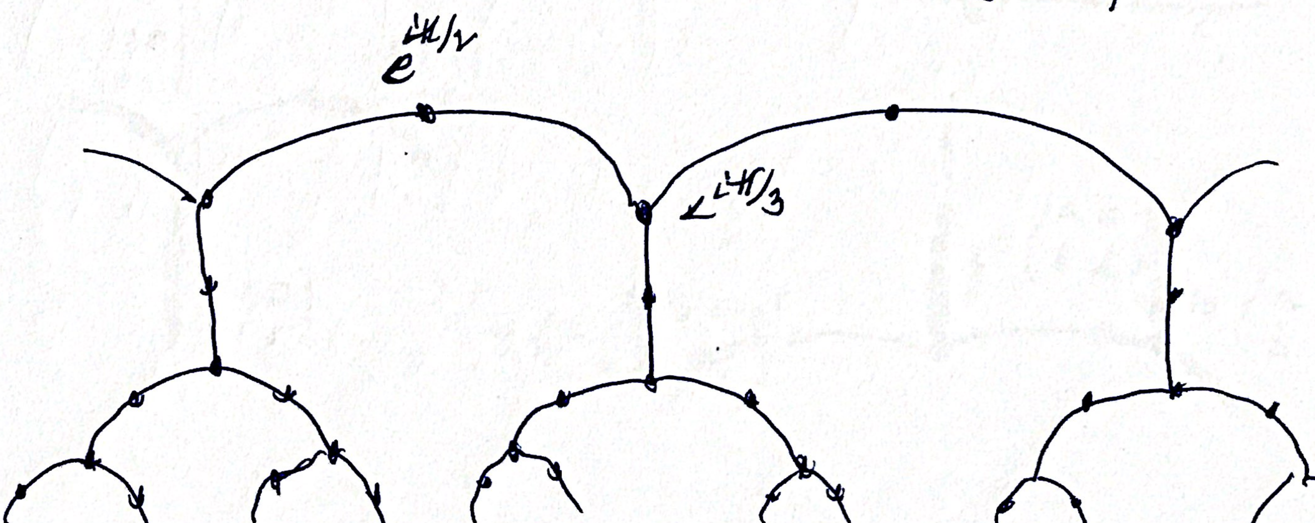
and $SL_2(\mathbb{Z})$ acts on $\mathfrak{H}_1$ by

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} z = \frac{az+c}{bz+d}$$

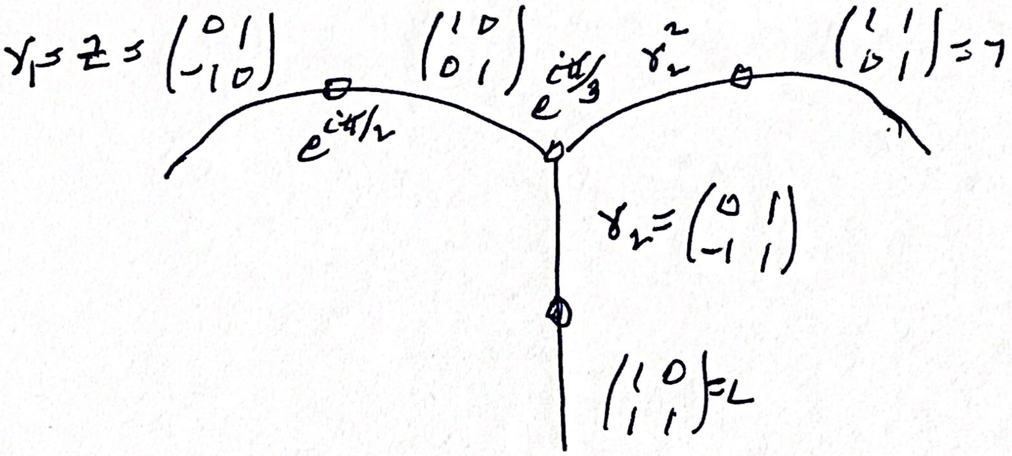
Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially this is an action of $PSL_2(\mathbb{Z})$. This action permutes

$$\left\{\begin{array}{c}\text{fundamental}\\ \text{regions}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{double cosets}\\ \text{in } \Gamma\backslash G/K\end{array}\right\}$$

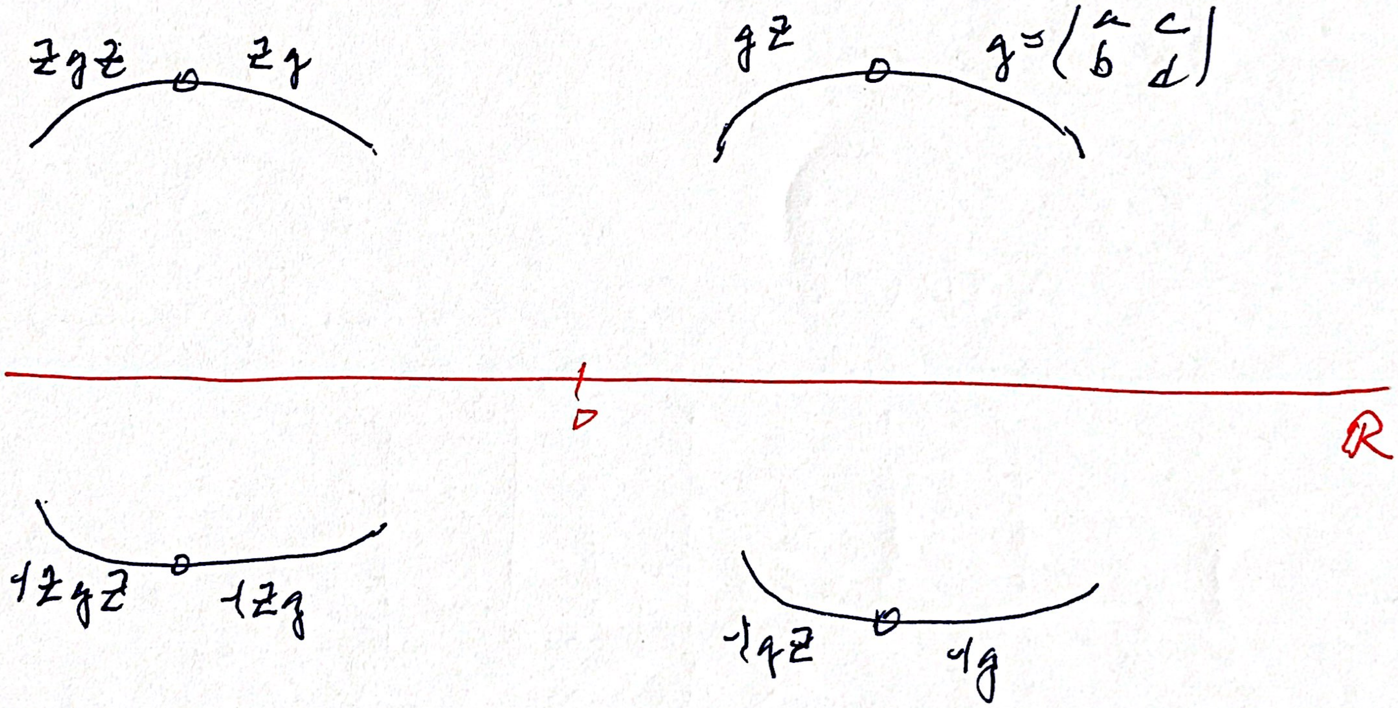




Generators: Label the trivalent tree



$SL_2(\mathbb{Z})$ is 8 copies of $SL_2(\mathbb{Z}_{\geq 0})$



$SL_2(\mathbb{Z}_{\geq 0})$ is a free monoid on two generators