c-functions

$y_1, \ldots, y_n$ are variables and $q, t^{\frac{1}{2}} \in \mathbb{C}^\times$.

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ the evaluation

$ev_\mu^t : \mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}] \to \mathbb{C}$ is given by

$$ev_\mu^t(y_i) = q^{-\mu_i} t^{-(v_\mu(i) - 1) + \frac{1}{2}(n-1)}$$

where $v_\mu \in S_n$ is minimal length such that $v_\mu \mu$ is weakly increasing.

An n-periodic permutation is a bijection $w : \mathbb{Z} \to \mathbb{Z}$ with $w(i+n) = w(i) + n$.

Let
$$Inv(w) = \left\{ (i,j) \;\middle|\; \begin{matrix} i \in \{1, \ldots, n\}, \; j \in \mathbb{Z} \\ i < j \text{ and } w(i) > w(j) \end{matrix} \right\}$$

For $i \in \{1, \ldots, n\}$ and $j \in \mathbb{Z}$,

$$y_{j+n} = q^{-1} y_j \quad \text{and} \quad c_{ij}^y = \frac{1 - t y_i y_j^{-1}}{1 - y_i y_j^{-1}}$$

Let

$$\ell(w) = \# Inv(w) \quad \text{and} \quad c_w^y = \prod_{(i,j) \in Inv(w)} c_{ij}^y .$$

Macdonald polynomials

Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$. The electronic Macdonald polynomial $E_\mu(q,t)$ is determined by

(a) $E_{(0,\dots,0)}(q,t) = 1$

(b) $E_{(\mu_1 - 1, \dots, \mu_n - 1)}(q,t) = (x_1 \cdots x_n)^{-1} E_{(\mu_1, \dots, \mu_n)}(q,t)$

(c) $E_{(\mu_n + 1, \mu_1, \dots, \mu_{n-1})}(q,t) = q^{\mu_n} x_1 E_{(\mu_1, \dots, \mu_n)}(q,t)$

(d) If $\mu_i > \mu_{i+1}$ then

$$E_{s_i\mu}(q,t) = (\partial_i x_i - t x_i \partial_i + r_\mu(q,t)) E_\mu(q,t)$$

where $s_i\mu = (\mu_1, \dots, \mu_{i+1}, \mu_i, \dots, \mu_n)$

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}$$ with $(s_i f)(x_1, \dots, x_n) = f(x_1, \dots, x_{i+1}, x_i, \dots x_n)$

$$r_\mu(q,t) = \frac{(1-t)\, q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}$$

Bosonic $P_\lambda(q,t)$ and Fermionic $A_{\lambda+\rho}(q,t)$

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n$ and

$\rho = (n-1, n-2, \dots, 2, 1, 0)$ and

$$c^{X^{-1}}_{w_0} = \prod_{i<j} \frac{1 - t x_i^{-1} x_j}{1 - x_i^{-1} x_j}.$$

$W_\lambda(t) = ev_0^t(c^Y_{w_0 v_\lambda})$ where $w_0(i) = n-i$.

$$P_\lambda(q,t) = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w(c^{X^{-1}}_{w_0} E_\lambda)$$

$$A_{\lambda+\rho}(q,t) = c^{X^{-1}}_{w_0} \sum_{w \in S_n} (-1)^{\ell(w_0 w)} w(E_{\lambda+\rho}).$$

E-expansions

$$P_\lambda(q,t) = \sum_{z \in W^\lambda} ev^t_{z\lambda}(c^Y_{v_\lambda z}) E_{z\lambda}(q,t)$$

$$A_{\lambda+\rho}(q,t) = \sum_{z \in S_n} (-1)^{\ell(w_0 z)} ev^t_{z(\lambda+\rho)}(c^{Y^{-1}}_{w_0 z}) E_{z(\lambda+\rho)}(q,t)$$

where

$W_\lambda = \mathrm{Stab}_{S_n}(\lambda)$ and

$W^\lambda = \{$minimal length representatives of cosets in $S_n/W_\lambda\}$

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Principal specializations

n-periodic permutations t_μ and u_μ given by

$t_\mu(i) = i + n\mu_i$, for $i \in \{1, \dots, n\}$

$t_\mu = u_\mu v_\mu$.

When $x_1 = 1, x_2 = t, \dots, x_n = t^{n-1}$ then

$$E_\mu(q,t) = t^{\frac{n-1}{2}|\mu| - \ell(t_\mu)} \, ev_0^t\left(c^{Y^{-1}}_{u_\mu}\right)$$

$$P_\lambda(q,t) = t^{\frac{n-1}{2}|\lambda|} \, ev_0^{t^{-1}}\left(c^{Y^{-1}}_{t_\lambda}\right)$$

$$A_{\lambda+\rho}(q,t) = 0.$$

where $|\mu| = \mu_1 + \dots + \mu_n$

Inner products

For $f_1, f_2 \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$,

$$(f_1, f_2)_{q,t} = ct\left(\frac{f_1 \overline{f_2}}{c_\infty^X \, c_0^X \, c_\infty^{X^{-1}}}\right)$$

where

$\overline{f}(x_1, \dots x_n; q, t) = f(x_1^{-1}, \dots x_n^{-1}; q, t)$

ct denotes 'constant term'

$$c_\infty^X = \prod_{i<j} \prod_{r=1}^{\infty} \frac{1 - t q^r x_i x_j^{-1}}{1 - q^r x_i x_j^{-1}} \qquad c_\infty^{X^{-1}} = \prod_{i<j} \prod_{r=1}^{\infty} \frac{1 - t q^r x_i^{-1} x_j}{1 - x_i^{-1} x_j}$$

$$c_0^X = \prod_{i<j} \frac{1 - t x_i x_j^{-1}}{1 - x_i x_j^{-1}}.$$

Orthogonality If $\lambda \neq \mu$ then

$(E_\lambda, E_\mu)_{q,t} = 0, \quad (P_\lambda, P_\mu)_{q,t} = 0, \quad (A_{\lambda+\rho}, A_{\mu+\rho})_{q,t} = 0.$

Reductions

$$(E_\mu, E_\mu)_{q,t} = ev_0^t\left(c_{u_\mu}^Y c_{u_\mu}^{Y^{-1}}\right) \cdot (1,1)_{q,t}$$

$$(P_\lambda, P_\lambda)_{q,t} = \frac{W_0(t)}{W_\lambda(t)} t^{\ell(v_\lambda)} ev^t\left(c_{v_\lambda}^Y\right) (E_\lambda, E_\lambda)_{q,t}$$

$$(A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t} = ev_{\lambda+\rho}^t\left(\frac{c_{w_0}^{Y^{-1}}}{c_{w_0}^Y}\right) \cdot (P_{\lambda+\rho}, P_{\lambda+\rho})_{q,t}$$

Shift

$$\frac{(P_\lambda(q,qt),P_\lambda(q,qt))_{q,qt}}{(P_{\lambda+\rho}(q,t),P_{\lambda+\rho}(q,t))_{q,t}} = \frac{W_0(qt)}{W_0(t)}\, ev^{t}_{\lambda+\rho}\left(t^{\ell(w_0)}\frac{c^{y^{-1}}_{w_0}(t)}{c^{y}_{w_0}(t)}\right)$$

Norm conjectures/theorems

$$(P_\lambda(q,q^k),P_\lambda(q,q^k))_{q,q^k} = W_0(q^k)\prod_{r=0}^{k-1} ev^{q^r}_{\lambda+(k-r)\rho}\left(q^{r\ell(w_0)}\frac{c^{y^{-1}}_{w_0}(q^r)}{c^{y}_{w_0}(q^r)}\right)$$

and

$$(P_\lambda(q,t),P_\lambda(q,t))_{q,t} = W_0(t)\, ev_{q^\lambda t^\rho}\left(\frac{1}{c^{y}_{w_0}(t)\,c^{y}_{w_0}(t)}\right) ev_{q^\lambda t^{-\rho}}\left(\frac{1}{c^{y^{-1}}_{w_0}(t^{-1})}\right)$$

where

$$ev_{q^\lambda t^\rho}(y_i) = q^{\lambda_i} t^{n-j}$$