Murphy elements and Casimirs

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Group algebra of S_k: $\mathbb{C}S_k$

Murphy's $M_j = \sum_{i=1}^{j-1} s_{ij}$

s_{ij} transposition switching i and j.

$d = M_1 + \cdots + M_k = \sum_{1\le i<j\le k} s_{ij}$ is in $Z(\mathbb{C}S_k)$

The enveloping algebra $U\mathfrak{gl}_n$

Generators: E_{ij} for $i,j \in \{1,\ldots,n\}$

Relations: $E_{ij}E_{k\ell} = E_{k\ell}E_{ij} + \delta_{jk}E_{i\ell} - \delta_{\ell i}E_{i\ell}$

Casimir: $\kappa = \sum_{i,j=1}^{n} E_{ij}E_{ji}$ is in $Z(U\mathfrak{gl}_n)$

The module $V^{\otimes k}$: $U\mathfrak{gl}_n$ acts by

$E_{ij}(v_1\otimes\cdots\otimes v_k) = \sum_{\ell=1}^{k} v_1\otimes\cdots\otimes v_{\ell-1}\otimes E_{ij}v_\ell\otimes v_{\ell+1}\otimes\cdots\otimes v_k.$

$\mathbb{C}S_k$ acts by

$w(v_1\otimes\cdots\otimes v_k) = v_{w^{-1}(1)}\otimes\cdots\otimes v_{w^{-1}(k)}$

If $V = \mathbb{C}^n$ then, as operators on $V^{\otimes k}$

$\kappa = d$.

Transvections and Hecke algebras

Group algebra of $G = GL_n(\mathbb{F}_q)$: $\mathbb{C}G$.

$\mathcal{C}$ the conjugacy class of $\begin{pmatrix} 1 & 1 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$

A <u>transvection</u> is $x \in \mathcal{C}$.

$$C = \sum_{x \in \mathcal{C}} x \quad \text{is in } Z(\mathbb{C}G).$$

The Hecke algebra H

$$H = \mathrm{span}\{T_w \mid w \in S_n\}, \quad \text{where}$$

$$T_w = \frac{1}{|B|} \sum_{x \in BwB} x, \qquad B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subseteq G$$

<u>The module 1_B^G</u>: For $g \in G$ let

$$v_g = \frac{1}{|B|} \sum_{x \in gB} x \quad \text{and} \quad 1_B^G = \mathrm{span}\{v_g \mid g \in G\}$$

$\mathbb{C}G$ acts by left multiplication on 1_B^G
H acts by right multiplication on 1_B^G.

Let

$$D = (q-1) \sum_{1 \le i < j \le n} q^{n-1-(j-i)} T_{s_{ij}}.$$

As operators on 1_B^G:

$$C = D.$$

The elements D_γ

For $\mu_1, \dots, \mu_\ell \in \mathbb{Z}_{>0}$ with $\mu_1 + \dots + \mu_\ell = n$

$\mathcal{C}_\mu$ the conjugacy class of $u_\mu = \begin{pmatrix} \left.\boxed{\begin{matrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & 1 \end{matrix}}\right\}\mu_1 & & 0 \\ & \ddots & \\ 0 & & \overbrace{\boxed{\begin{matrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & 1 \end{matrix}}}^{\mu_\ell} \end{pmatrix}$

$$C_\mu = \sum_{x \in \mathcal{C}_\mu} x \quad \text{is in} \quad Z(\mathbb{C}G)$$

Find $D_\mu \in Z(H)$ that acts same way on 1_B^G.

$$G = \bigsqcup_{\substack{\text{conj.} \\ \text{classes}}} \mathcal{C}_\gamma \quad \text{and} \quad G = \bigsqcup_{w \in S_n} BwB$$

let
$$D_{\gamma,w} = |\mathcal{C}_\gamma \cap BwB|$$

and
$$D_\gamma = \sum_{w \in S_n} D_{\gamma,w}\, q^{-\ell(w)} T_w$$

Then $D_\gamma \in Z(H)$ and, as operators on 1_B^G,

$$C_\gamma = D_\gamma.$$

The matrix D

The values $D_{\gamma,w}$ are determined by

$D_{\gamma,\nu}$ where ν is minimal length in its conjugacy class in S_n.

Let $$D = (D_{\mu,\nu})_{\substack{\mu \in \mathcal{C}^{unip} \\ \nu \in \mathcal{W}}}$$

$\mathcal{C}^{unip}$ is an index set for conjugacy classes of unipotent elements in G

$\mathcal{W}$ is an index set for conjugacy classes in S_n

Can you compute D?

$$Q_\mu(x_1,\ldots,x_n;t) = b_\mu(t) P_\mu(x_1,\ldots,x_n;t)$$

the dual Hall-Littlewood symmetric function

m_ν the monomial symmetric function

$$Q_\mu = \sum_\nu a_{\mu\nu}(t) m_\nu.$$

Then $$D_{\mu,\nu} = \frac{q^{n+n(\mu)}}{(1-q)^{\ell(\nu)}} a_{\mu,\nu}(q^{-1})$$

$$= \frac{q^{n+n(\mu)}}{(1-q)^{\ell(\nu)}} \sum_{\substack{T \text{ column strict} \\ \text{shape } \mu \\ \text{content } \nu}} wt_q(T).$$