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Hecke algebras

G a group, $\Gamma \subseteq G$ a subgroup.

W a set of reps of cosets in $\Gamma\backslash G/\Gamma$

$$G = \bigsqcup_{w\in W} \Gamma w \Gamma$$

$\mathbb{C}[G] = \mathrm{span}\{g \in G\}$ is the group algebra

$$T_w = \frac{1}{|\Gamma|}\sum_{x\in\Gamma w\Gamma} x, \quad \text{for } w \in W.$$

The <u>Hecke algebra of the pair $G \supseteq \Gamma$</u> is

$$H = \mathrm{span}\{T_w \mid w \in W\}$$

<u>Actions of H</u>

M is a set with a G-action

$\mathcal{F}$ a set of reps of orbits in $\Gamma\backslash M$

$$M = \bigsqcup_{t\in\mathcal{F}} \Gamma t$$

$\mathbb{C}[M] = \mathrm{span}\{m \in M\}$ contains

$$v_t = \frac{1}{|\Gamma|}\sum_{m\in\Gamma t} m, \quad \text{for } t \in \mathcal{F}$$

Then H acts (by left multiplication) on

$$\mathbb{C}[\Gamma\backslash M] = \mathrm{span}\{v_t \mid t \in \mathcal{F}\}.$$

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Our example

$$G = GL_2(\mathbb{Q}) \text{ and } \Gamma = SL_2(\mathbb{Z})$$

Then

$$G = \bigsqcup_{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \in A} \Gamma \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma, \quad \text{where}$$

$$A = \left\{ \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \middle| \begin{array}{l} d_1, d_2 \in \mathbb{Q} \\ d_1\mathbb{Z} \subseteq d_2\mathbb{Z} \end{array} \right\}$$

and

$$\Gamma \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma = \bigsqcup_{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{F}} \Gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad \text{where}$$

$$\mathcal{F} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \begin{array}{l} a, b, d \in \mathbb{Q},\ a > 0,\ ad = d_1 d_2 \\ \gcd(a, b, d) = d_2 \end{array} \right\}$$

So H has basis $\left\{ T_{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}} \middle| \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \in A \right\}$

Remark on decompositions

$$G = GL_n(\mathbb{R}) \text{ and } K = O_n(\mathbb{R})$$

$$B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subseteq G, \quad A = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \middle| \begin{array}{l} a_1, \ldots, a_n \in \mathbb{R}_{>0} \\ a_1 \geq \cdots \geq a_n \end{array} \right\},$$

Then

$$G = KB \text{ and } G = KAK.$$

This Hecke algebra is commutative

$$\mathbb{C}[GL_2(\mathbb{Q})] \longrightarrow \mathbb{C}[GL_2(\mathbb{Q})] \qquad \text{has} \quad (gh)^t = h^t g^t$$
$$g \longmapsto g^t \qquad\qquad (g^t)^t = g.$$

Then

$$T^t_{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}} = \frac{1}{|\Gamma|} \sum_{x \in \Gamma \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma} x^t = \frac{1}{|\Gamma|} \sum_{y \in \Gamma \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma} y = T_{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}}.$$

and

$$T_{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}} T_{\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}} = T^t_{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}} T^t_{\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}} = \left(T_{\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}} T_{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}} \right)^t$$
$$= T_{\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}} T_{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}}$$

The Hecke operators $T(n)$, for $n \in \mathbb{Z}_{>0}$

$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{Z} \\ ad-bc = n \end{matrix} \right\}$$

$$= \bigsqcup_{\substack{d_1, d_2 \in \mathbb{Z}_{>0} \\ d_1\mathbb{Z} \supseteq d_2\mathbb{Z} \\ d_1 d_2 = n}} SL_2(\mathbb{Z}) \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} SL_2(\mathbb{Z})$$

$$= \bigsqcup_{\substack{a,d \in \mathbb{Z}_{>0} \\ ad = n}} \bigsqcup_{b=0}^{d-1} SL_2(\mathbb{Z}) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

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The <u>Hecke operators</u> $T(n)$ are

$$T(n) = \frac{1}{|\Gamma|} \sum_{x \in \Delta_n} x = \sum_{\substack{d_1 d_2 = n \\ d_1, d_2 \in \mathbb{Z}_{>0}}} T_{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}}$$

$$= \frac{1}{|\Gamma|} \sum_{ad=n} \sum_{b=0}^{d-1} \sum_{\gamma \in SL_2(\mathbb{Z})} \gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

<u>The action on cusp forms</u>

$$(T(n)f)(z) = \sum_{ad=n} \sum_{b=0}^{d-1} \left(\frac{a}{d}\right)^{k/2} f\left(\frac{az+b}{0z+d}\right)$$

for $f \in S_k(\Gamma)$ (space of cusp forms), where a cusp form of weight k is
$f : \mathcal{H} \to \mathbb{C}$ where $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$
and (a) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$,

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

(b) f is holomorphic at ∞
(c) f vanishes at ∞.

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The product $T(m)T(n)$

$$T(m)T(n) = \sum_{\gamma\in SL_2(\mathbb{Z})} \sum_{xz=m} \sum_{y=0}^{m-1} \sum_{ad=n} \sum_{b=0}^{n-1} \gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$

$$= \sum_{\gamma\in SL_2(\mathbb{Z})} \sum_{xz=m} \sum_{y=0}^{m-1} \sum_{ad=n} \sum_{b=0}^{n-1} \gamma \begin{pmatrix} ax & ay+bz \\ 0 & dz \end{pmatrix}$$

$$= T(mn) \quad \text{if } \gcd(m,n)=1.$$

The product $T(p)T(p^r)$

Let $r\in\mathbb{Z}_{>0}$ and let $p\in\mathbb{Z}_{>0}$ be prime.

$$T(p)T(p^r) = \sum_{\gamma\in SL_2(\mathbb{Z})} \sum_{xz=p} \sum_{y=0}^{p-1} \sum_{ad=p^r} \sum_{b=0}^{d-1} \gamma \begin{pmatrix} ax & ay+bz \\ 0 & dz \end{pmatrix}$$

$$= \sum_{\gamma\in SL_2(\mathbb{Z})} \sum_{st=p^{r+1}} \sum_{u=0}^{t-1} \gamma \begin{pmatrix} s & u \\ 0 & t \end{pmatrix}$$

$$+ p \sum_{\gamma\in SL_2(\mathbb{Z})} \gamma \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} u & l \\ 0 & w \end{pmatrix}$$

$$= T(p^{r+1}) + p\, T_{\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}} T(p^{r-1}).$$